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Deformations of $\mathcal{N} = 4$ SYM and integrable spin chain models

David Berenstein[*§] **and Sergey A. Cherkis** [*]

∗ *School of Natural Sciences, Institute for Advanced Study, Princeton, NJ 08540*
§ *Department of Physics, University of California at Santa Barbara, CA 93106*

Abstract: Beginning with the planar limit of $\mathcal{N} = 4$ SYM theory, we study planar diagrams for field theory deformations of $\mathcal{N} = 4$ which are marginal at the free field theory level. We show that the requirement of integrability of the full one loop dilatation operator in the scalar sector, places very strong constraints on the field theory, so that the only soluble models correspond essentially to orbifolds of $\mathcal{N} = 4$ SYM. For these, the associated spin chain model gets twisted boundary conditions that depend on the length of the chain, but which are still integrable. We also show that theories with integrable subsectors appear quite generically, and it is possible to engineer integrable subsectors to have some specific symmetry, however these do not generally lead to full integrability. We also try to construct a theory whose spin chain has quantum group symmetry $SO_q(6)$ as a deformation of the $SO(6)$ R-symmetry structure of $\mathcal{N} = 4$ SYM. We show that it is not possible to obtain a spin chain with that symmetry from deformations of the scalar potential of $\mathcal{N} = 4$ SYM.
We also show that the natural context for these questions can be better phrased in terms of multi-matrix quantum mechanics rather than in four dimensional field theories.

Contents

1. Introduction

With the advent of the AdS/CFT correspondence [1, 2, 3] the study of four dimensional conformal field theories has generated a new interest in the past few years. The correspondence makes an equivalence between string backgrounds of the form $AdS_5 \times W$ in the presence of RR fluxes and conformal field theories in four dimensions which have a large N limit, where W is some compact geometry. All of the examples seem to require a gauge field theory on the boundary. We can even generalize the correspondence to other large N theories which are not conformal, so long as we replace the AdS geometry by a more general background. In particular this suggest a new approach to explore the large N limit of QCD.

In an ideal situation we would have a solvable four dimensional theory where we can see the correspondence and establish the equivalence between all the string states

in $AdS_5 \times W$ and the associated conformal field theory observables. Our main obstacle to see this correspondence in explicit details is the lack of tools to make reliable calculations. From the field theory point of view, for the most part we are restricted to perturbative calculations. Thus we are only able to study the field theory if we have setups where we have a free field theory limit with a tunable 't Hooft coupling $\lambda = g_{YM}^2 N$ which can be taken small, and study the problem order by order in perturbation theory. Similarly, calculability on the string geometry usually requires us to have a weakly curved background, where we can expand systematically around a type IIB supergravity solution where W is a very large manifold of dimension 5. The radius of curvature of AdS and W, R is roughly given by

$$R \sim \sqrt[4]{\lambda} \tag{1.1}$$

so the calculability of the spectrum takes us to large values of R, which translate into large values of the 't Hooft coupling λ. Comparison of both sides of the correspondence places us into a strong/weak coupling duality for the 't Hooft parameter.

In this setup we also have to worry about the strings being free and not interacting. The string coupling constant in ten dimensions is roughly given by $g_{closed} \sim g_{ym}^2$, and the gravitational constant when we reduce to the five dimensional AdS geometry is roughly $1/N^2$, so making λ small does not make the strings interact more, and in fact they are free. However, the worldsheet sigma model becomes strongly coupled, so all results are plagued by having large α' corrections.

Of all these field theories, $\mathcal{N} = 4$ SYM theory with gauge group $SU(N)$ is special. The supergravity dual geometry is very simple, namely $AdS_5 \times S^5$ [1]. The large amount of supersymmetry ensures that the theory is essentially finite, moreover the representation theory of the symmetry algebra dramatically simplifies the spectrum of states because the representations carry a lot more states than in other field theories. Because of it's special nature, most of the work has centered on studying this one example.

The $\mathcal{N} = 4$ SYM in the $\mathcal{N} = 1$ terms has three chiral superfields X, Y, Z in the adjoint of $SU(N)$, and a superpotential proportional to $\mathrm{tr}(X[Y,Z])$, plus their coupling to the vector multiplet of $SU(N)$. The theory has an $SU(4) \sim SO(6)$ R-charge symmetry, and the superconformal group is identified with $SU(2,2|4)$.

The first tests of the correspondence involved states which do not get quantum corrections when we turn on the gauge coupling. These states are half BPS states and are protected by supersymmetry. Witten [2] showed that these states match the spectrum of supergravity fluctuations if we identify individual quanta on AdS with single trace operators built from scalars which are totally symmetric and traceless. All these are descendants under the superconformal group of the single state

$$\mathrm{tr}(Z^J), \tag{1.2}$$

for all values of $J = 2, 3, \ldots$, so long as we take $N \to \infty$ first.

The next breakthrough in the description of the theory beyond BPS states came from understanding that one can fabricate string states which are near BPS by taking a plane wave limit on $AdS_5 \times S^5$ [4]. Coupled with the fact that the plane wave geometries have a solvable string spectrum [5], it was possible to give expressions of the dimensions of operators which interpolated between weak and strong coupling λ. It turned out that since the operators were nearly BPS, the effective expansion parameter was λ/J^2, so one can extrapolate results from small λ to large λ if one scaled J appropriately, and indeed, all of these results could be reproduced up to one loop and matched with a particular list of operators similar to (1.2) where a few of the Z's are replaced by other fields which are treated as impurities. This result suggested that the system should be treated as some sort of a spin chain model where only planar diagrams are considered.

Two more developments came afterwards which made the $\mathcal{N} = 4$ SYM theory a lot more interesting. First, Minahan and Zarembo [6] showed that a sector of the single trace operators made of scalars gives rise to an integrable $SO(6)$ spin chain model at one loop, a result which was later generalized to all single trace operators with the full $SU(2,2|4)$ superconformal group by Beisert and Staudacher [7, 8] using results from anomalous dimension calculations in QCD [9, 10, 11], where another integrable spin chain was found for a restricted set of operators, and which resembled the $XXX_{-1/2}$ spin chain. The second development was the realization by Bena, Polchinski and Roiban [12] that the Green-Schwarz sigma model for strings on $AdS_5 \times S^5$ leads to a integrable sigma model.

It was then conjectured by Dolan, Nappi and Witten [13] that these two results can be tied together because they both lead to the same type of symmetry structure based on a Yangian, so that there should be an integrable structure which exists for all values of the 't Hooft coupling λ . This symmetry would give us a correspondence between the perturbative description of $\mathcal{N} = 4$ SYM theory and all the string states of the type IIB string on $AdS_5 \times S^5$ if we follow it carefully.

Progress along these lines has been very fruitful in the last few months. Higher loop computations have been performed and integrability seems to persist. See [14, 15] and references therein for a more thorough review of these developments. Also, semiclassical string configurations have been studied in a lot of detail, and dual candidate states have been identified in terms of the spin chain model via the thermodynamic Bethe ansatz, see [16, 17] and references therein.

Given the above successes for $\mathcal{N} = 4$ SYM theory, it is important to consider less supersymmetric cases, not just for their applications to QCD, but also as a way to determine how string theory in more general RR backgrounds behaves. This will also serve to determine how special is the $AdS_5 \times S^5$ geometry as a RR target space for string theory.

Our objective for the paper is to follow exactly this path: to study other conformal field theories in four dimensions and to determine if they will be integrable or

not. The original intent was to look for solvable models that can serve as a guide to study the manifold W for the $AdS_5 \times W$ geometry for a case which is not already known. Indeed, as a string background, W does not have to be a large manifold where supegravity is valid, but it can be a consistent background for string theory much the same way that a Gepner model or other exactly solvable CFT on the string worldsheet theory is considered as a target geometry for string theory. This is so even though all of the target space features and volumes are of stringy size and are not particularly geometric. The fact that we have the spectrum of states is the key factor in determining the target space properties.

Amongst all of these, we can study orbifolds of $AdS_5 \times S^5$. These have a known dual, and from this point of view they are interesting geometries. However, from the integrable structure point of view they are not teaching us very much at all.

A sigma model of a string theory on an orbifold is essentially the same sigma model for the string theory in the original theory. The only new ingredient is that some states are projected out, and that there is a twisted sector of states. The new sector affects the periodicity conditions of the sigma model on a circle, but they should not affect the local integrability properties of the model. From this point of view, the sigma model prediction tells us that the failure of integrability might be in the boundary conditions, if there is any at all. Similarly, we can consider theories where we modify the $\mathcal{N} = 4$ SYM theory by adding open strings, and checking if the boundary conditions are integrable. This corresponds to adding D-branes to $AdS_5 \times S^5$. Again, the integrability of the bulk of the string is not in question, but only the boundary conditions that it is subject to are. In the framework of this paper these type of models will be considered "trivial" in that they don't modify the local structure of the integrable sigma model, but only the boundary conditions. These issues have been explored previously in [18, 19, 20, 21].

For more general theories, most of the claims along these lines prove integrability for a subsector of the theory up to one loop order. Although this fact is interesting, it is not the same as proving integrability of the full model up to one loop. This is the type of integrability we will be looking for.

Given that there is such a large list of conformal field theories, we concentrate on studying a simple class of theories obtained by a special class of marginal deformations of $\mathcal{N} = 4$ SYM, so that we can study theories closely related to $\mathcal{N} = 4$. Some of these are known to lead to four dimensional superconformal field theories [22]. Other deformations we study look marginal at the free field level and break supersymmetry, and we will be asking wether it is possible to obtain integrability to one loop order or not.

For simplicity, we also want to deal with situations where a large group of symmetries of $\mathcal{N} = 4$ is unbroken: we want to have the conformal group of AdS_5 and the Cartan of the R-symmetry group unbroken. This will simplify the spin chain analysis and will provide us with models that can be shown to be integrable or not

by using the Bethe ansatz.

Some of the theories that are accessible this way turn out to be orbifolds of $\mathcal{N}=4$ SYM theories [23, 24, 25, 26] , so at least we are guaranteed some success in looking for integrable models. Moreover we can explore how the boundary conditions of the string determine different backgrounds, some of which will not be orbifolds, but still are closely related to them. Some of these were found by Roiban [37], where he also described a way to engineer field theories with a subsector which is integrable. Here we explore much more deeply this problem and we find that there are additional constraints on the spin chain models one can write due to properties of Feynman diagrams, so that this engineering is not guaranteed to produce a reasonable field theory potential with a larger integrable sector.

Our quest for solutions to the integrability problem resulted in no new models within a very interesting class of theories which are not "trivial" in the sense we described above, even though one can find large integrable subsectors in some of them. It turns out that while it is possible to generically find subsectors which are integrable, as we consider more general states the constraints imposed by integrability become much more powerful and ultimately the models in question ends up being either "trivial". Although one can phrase these results in the paper as some form of "no-go" theorem, we have found many interesting results along the way that are worthwhile on their own. In particular, at the end of the paper we find that the natural setup for the correspondence between integrable spin chains and large N theories is via multi-matrix quantum mechanics, rather than four dimensional field theories (these are after all a particular examples of multi-matrix quantum mechanics with an infinite number of matrices). Once in the matrix model setup, one can find a correspondence between arbitrarily local spin chain models and large N multi matrix models, and in particular one can engineer integrable multi-matrix models in the large N limit. Hopefully these techniques will prove of further use, maybe even in the study of the $\mathcal{N}=4$ SYM theory itself.

2. Supersymmetric marginal deformations of $\mathcal{N}=4$ SYM

The $\mathcal{N}=4$ SYM theory has a moduli space of supersymmetric marginal deformations of dimension 3 [22]. Their Lagrangians, for gauge group $U(N)$ are characterized by the following superpotential

$$W(\phi) = A(\mathrm{tr}(XYZ) - q\mathrm{tr}(XZY) + h\mathrm{tr}(X^3+Y^3+Z^3)) \tag{2.1}$$

Here A is a normalization factor which can be changed if we also change the kinetic terms of the fields X, Y, Z. There is also a gauge coupling, so once the normalization of the fields is chosen, A is a particular function of q, h, g_{YM}, which can be determined by perturbation theory. The above theories have a Z_3 symmetry $X \to Y \to Z \to X$,

and a $U(1)_R$ charge which is part of the superconformal algebra. The $N = 4$ theory appears when $q = 1, h = 0$. In our paper we will only deal with the case $h = 0$, so we set it to this value from now on.

The classical chiral ring of the theory is independent of A. The theory can have a moduli space of vacua which depends on the rank of the gauge group and on the couplings. This has been explored in detail in [26]. In this moduli space of vacua the vacuum characterized by $X = Y = Z = 0$ is the origin, and it is here where the theory has an unbroken conformal invariance.

The theories can be studied in the large N limit, and in light of the AdS/CFT correspondence one might try to understand if there is a supergravity background dual to these theories. This requires to scale all terms of the Lagrangian with a uniform factor of N outside, and to have coefficients independent of N in all terms in the Lagrangian. Also, one does not take into account all Feynman diagrams, but only planar diagrams. The limit $N \to \infty$ keeping all other coefficients in the Lagrangian fixed is the 't Hooft limit of the theory. The theory is then perturbative in the 't Hooft couplings $\lambda = g^2 N$, $\lambda' = A/N$, with h, q fixed complex numbers. Conformal invariance then produces a relation between λ' as a function of λ, so one can analyze the full theory as perturbation theory in the 't Hooft coupling λ. At the $N = 4$ level, $\lambda' \sim 1/\lambda$.

For the maximally supersymmetric theory, the dual background is $AdS_5 \times S^5$. For $h = 0$ and q a primitive n-th root of unity the dual supergravity background is given by the orbifold $AdS_5 \times S^5/Z_n \times Z_n$, which corresponds to a very different geometry which is not a small deformation of $AdS_5 \times S^5$. For other values of q very little is known, except for perturbations of q around 1, which can be identified in the supergravity dual of $AdS_5 \times S^5$ [27].

The AdS/CFT correspondence tells us that the dual background should be a string-theory compactification on $AdS_5 \times X$. However, X does not have to be geometrical in the supergravity sense. It can just as well be a string background with characteristic curvatures and features of order of the string scale, even in the large t' Hooft coupling limit (X can have singularities or small circles). Solving the large N theory would be equivalent to understanding what X is. For q close to some special values it might have good geometric interpretation. In that case one might try to recover the geometry of X along the lines of [28]. In this paper we will think of solvability as being able to find the full spectrum of strings on $AdS_5 \times X$. This should be thought of as being equivalent to solving a CFT with RR backgrounds exactly, so it can be understood as a RR Gepner model.

The observables of the theory are the correlation functions of local gauge invariant operators of the theory. Via radial quantization in the Euclidean theory, every such operator corresponds to a state of the theory when it is compactified on $S^3 \times R$. The energy of the state is the eigenvalue with respect to the generator of scale transformations: the dimension of the operator.

Restricting to planar diagrams and operators of fixed dimension in the free field theory as $N \to \infty$, the set of operators can be characterized by the number of traces in the operator, and planar diagrams do not change this number of traces. In particular, the spectrum of states becomes a Fock space with one oscillator for every single trace gauge invariant operator. Each one of these is interpreted as a single string state on the AdS/CFT dual.

We are interested in calculating the dimension of all of these local operators. In perturbation theory this amounts to calculating the planar anomalous dimension of the associated operator.

In the free field theory limit, all dimensions are integers or half-integers and there is a large degeneracy of dimensions. Thus, to first order in perturbation theory it is important to diagonalize the one loop effective Hamiltonian on the basis of states with equal energy in the free field limit. If we ignore non planar diagrams, this reduces to the problem of diagonalizing a particular spin chain Hamiltonian with periodic boundary conditions determined by the interactions of the quantum field theory.

3. One loop anomalous dimensions as a spin chain Hamiltonian

Now, we will focus on the problem of finding the (planar) one loop anomalous dimensions of chiral operators for the q-deformation f $\mathcal{N} = 4$ SYM theory. ¿From the CFT point of view this is natural, as there are short representations of the superconformal group in four dimensions which are chiral. These will have protected dimensions equal to the R-charge. Moreover, in [4] it was argued that there can be unprotected chiral operators which are almost BPS and with finite anomalous dimensions in the large $\lambda = g^2 N$ limit, so long as we scale the R-charge J as $\sqrt{\lambda}$, so these operators lie in an interesting class of operators. Technically, these are also simpler to understand, as the only contribution to their one-loop anomalous dimensions comes from the F-terms in the supersymmetric Lagrangian (the D-term contributions cancel against the photon exchange [30]), so the number of diagrams that need to be calculated is smaller. Also, two and three spin solutions of semiclassical string configurations in $AdS_5 \times S^5$ fall into this class of operators and have been studied extensively [16]

For the time being we will concentrate only on chiral operators built out of X, Y alone. These are of the form

$$tr(XXYXXXY\dots) \tag{3.1}$$

The cyclicity of the trace makes some identifications between different orderings of the X, Y fields. Modulo this identification, different words made out of the X, Y are orthogonal in the planar limit. We can implement the cyclicity condition at the

end, and work simply with periodic chains. The cyclicity condition can be obtained by summing any operator over all it's possible translates. In essence, removing the cyclic condition is tantamount to marking an initial letter in the cyclic word made out of X, Y. With this convention, we can label a letter by the position in the word in terms of the distance i from this marked letter, position of which we call 0.

We can map the vector space spanning these operators to the vector space of a spin 1/2 chain $\otimes_{i=0}^{n-1}(|0> \oplus |1>)_i = \otimes_{i=0}^{n-1} V_i$ where we assign a zero whenever we find the letter X, and a 1 whenever we find the letter Y.

Moreover, it has been argued in [25, 26] that the operators $tr(X^m)$ are non-trivial elements of the chiral ring for all possible values of q, since all of these can get a vev on the moduli space of vacua. Moreover the deformation preserves a $U(1)^3$ symmetry of complex rotations of X, Y, Z separately. The operator $tr(X^n)$ is then the only chiral operator of dimension n with charge n under the $U(1)$ that rotates X. Since it is a chiral operator which is not trivial in the chiral ring, the anomalous dimension of the state $|00\dots 0) >$ is zero, and it can be used as a reference state, since we do not need to worry about mixing of the word X^n with other orderings of the fields.

Unitarity of the conformal field theory implies that the anomalous dimensions for chiral operators are positive. In the free field theory limit all of these operators are in small representations of the superconformal group, so it is interesting to ask how these dimensions depend on q to leading order in the g^2N expansion.

Planarity of the diagrams implies that to the leading order the matrix of anomalous dimensions receives contributions only from nearest neighbor interactions. Moreover, the $U(1)^3$ symmetry of the Lagrangian guarantees that the number of Y and X are preserved by the interactions in the Lagrangian, and that these words do not mix with other ones. These contributions can be read from the F-term Lagrangian

$$\delta L = \frac{N}{\lambda} tr((XY - qYX)(\bar{Y}\bar{X} - q^*\bar{X}\bar{Y})) \tag{3.2}$$

Here we have chosen the normalization of the kinetic term to be given by

$$\int \frac{N}{\lambda} \partial X \partial \bar{X}$$

With this convention the anomalous dimensions are proportional to λ. The proportionality constant can be read off from the OPE of δL and the operators $\mathcal{O}$

(and we are ignoring constant factors of order unity). These are

$$
\begin{array}{ll}
|00> \to |00> & 0 \\
|11> \to |11> & 0 \\
|10> \to |10> & qq^* \\
|01> \to |01> & 1 \\
|10> \to |01> & -q \\
|01> \to |10> & -q^*
\end{array}
$$

In 3.2 the $\bar{X}, \bar{Y}$ fields are interpreted as destruction operators for the fields X, Y, while X, Y are interpreted as creation operators. Given that we have an equal number of each, the Hamiltonian keeps the number of fields in an operator fixed, but can alter the order of the configuration (this is a spin exchange interaction in the spin chain model).

Also all contributions where the total number of spins up and down in a nearest neighbor pair differ from the initial and final state (keeping all others fixed) are zero. This is because of conservation of the $U(1)^2$ symmetry of the Hamiltonian.

Thus, the matrix of one-loop anomalous dimensions in this sector is given by the following periodic spin chain Hamiltonian

$$\frac{H}{\lambda'} = \sum_i \frac{1}{4}[(1-2\sigma_i^3)(1+2\sigma_{i+1}^3)+qq^*(1+2\sigma_i^3)(1-2\sigma_{i+1}^3)-q\sigma_i^-\sigma_{i+1}^+-q^*\sigma_i^+\sigma_{i+1}^- \quad (3.3)$$

In the above notation, σ_i^3 is one of the Pauli matrices for the spin associated to position i, and $\sigma_i^\pm = (\sigma^1 \pm \sigma^2)_i$. The periodicity of the boundary conditions is included when we make the identification $\sigma_n = \sigma_0$, and $\lambda' = \lambda/16\pi^2$ includes the numerical factors from the one loop integral. For our purposes the precise coefficient does not matter, just the relative coefficients from different terms in the effective Hamiltonian.

The reader can easily verify that $H|00\cdots>= 0$, and that $[H, \sum_i \sigma_i^3] = 0$. This verifies that our reference ground state $|00\cdots>$ is a chiral primary to one loop order. Also, if q is real the above Hamiltonian can be written in the following form

$$\frac{H}{\lambda'} = \sum_i \frac{1+q^2}{4} - (1+q^2)\sigma_i^3\sigma_{i+1}^3 - 2q\sigma_i^1\sigma_{i+1}^1 - 2q\sigma_i^2\sigma_{i+1}^2 \quad (3.4)$$

which is exactly the XXZ spin chain hamiltonian, which is well known to be integrable. Indeed, any nearest neighbor Hamiltonian on a spin 1/2 chain which preserves J_z is the XXZ spin chain in the presence of a constant magnetic field. For $q = 1$, this is the XXX spin chain hamiltonian, where there is an additional $SU(2)$ symmetry. This is the case that corresponds to $\mathcal{N} = 4$ SYM theory. After normalization to the form

$$A - \sum \sigma^1 \otimes \sigma^1 + \sigma^2 \otimes \sigma^2 + \Delta\sigma^3 \otimes \sigma^3 \quad (3.5)$$

with Δ the anisotropy parameter of the spin chain, we find that $\Delta = \frac{1+q^2}{2q} \geq 1$. With this condition the spin chain is ferromagnetic, and the ground state has all spins down or up, and corresponds to the Bethe reference state as described above. Usually in condensed matter systems it is more interesting to explore the theory where $\Delta < 1$ and the ground state is not the ferromagnet.

Now, let us look at the system for q given by any complex number $q = r\exp(i\theta)$. It is convenient to do the following (position dependent) change of basis on the spin chain

$$|0>_k = |\tilde{0}>_k; |1>_k = \exp(ik\theta)|\tilde{1}>_k \tag{3.6}$$

for $k = 0, \dots n$. The $\sigma^\pm$ matrices are related between these bases by a similarity transformation $\sigma^\pm_k = \exp(\pm ik\theta)\tilde{\sigma}^\pm_k$, and in the new basis, working on equation (3.3) we have a spin chain Hamiltonian that is given again by (3.4), where we substitute $q = r$, and all σ matrices by $\tilde{\sigma}$ matrices. In particular, as far as the spin chain Hamiltonian is concerned, all values of q related by complex phase rotations are equivalent. However, the complex phase θ makes its appearance as a change in the boundary condition along the chain, because we identify

$$|\tilde{1}>_0 = \exp(in\theta)|\tilde{1}>_n \tag{3.7}$$

The fact that the theory with different couplings can give rise to the same spin chain Hamiltonian is important. Notice that for $r = 1$ we can get the XXX spin chain, even though the original Lagrangian does not respect the $SU(2)$ symmetry. What really happens is that the boundary conditions do not respect the $SU(2)$ symmetry, so the field theory Lagrangian does not have to make the symmetry manifest. This symmetry is only present for planar diagrams, but not for the spectrum of the theory. It will become manifest locally on the spin chain once we do field redefinitions with position dependence, but the boundary conditions will still spoil the symmetry.

This change in the boundary conditions twists the XXZ spin chain model. These boundary conditions also result in a model which is soluble by the Bethe ansatz. Notice that the total phase accumulated depends on the length of the chain. Hence we find our promised result that the one loop anomalous dimensions for the q deformation of $\mathcal{N} = 4$ gives rise to the twisted XXZ spin chain model in the ferromagnetic regime.

Notice that the chain is periodic with usual periodicity of Eq.(3.7) whenever $\exp(in\theta) = 1$. This singles out the roots of unity as places which have many common features with the case $\theta = 0$. ¿From the field theory point of view, and when $r = 1$, these are exactly the orbifold points of the theory [23, 24]. Now it should be clear that at these points (for particular values of n) the physics should be very similar to the case where $\theta = 0$. These should correspond to the untwisted states of the orbifold theory. Now, for other values of n we get a total twist on the boundary condition, and this should be the case when we study the twisted sector of the orbifold theory.

The bulk of the work is now to find the spectrum of translationally invariant states of the spin chain. Since the change of basis between the two bases is position dependent, in the new basis of the spin chain, where the symmetries of the Hamiltonian can be easily identified, the translation operator is more complicated. Part of this program will be fulfilled in the next subsection.

3.1 Bethe ansatz for the twisted spin chain

The translation operator in the original basis just changes the configurations by sending $T : V_i \to V_{i+1}$ in the canonical identification between the V. In the new basis, $T|\tilde{0}>_i = |\tilde{0}>_{i+1}$ and $T|\tilde{1}>_i = \exp(i\theta)|\tilde{1}>_{i+1}$.

In particular T acts preserving the number of spins up and down in the $\tilde{V}_i$ basis. The additional phase with respect to the ground state acquired by a word with m spins up is $\exp(im\theta)$.

To solve the system with s spins up (impurities), we first change basis on the Hilbert space to a position basis, where we define $|\mathcal{G}\rangle = |000\dots 0>$ and the position vectors as $|x_1, \dots, x_s> = (\tilde{\sigma}^+)_{x_1} \dots (\tilde{\sigma}^+)_{x_s}|\mathcal{G}\rangle$. We choose the x_i so that $x_1 < x_2 < \dots x_s$, in order to have a one to one map between the states at each stage. The translation operator acts then by

$$T|x_1, \dots, x_s> = |x_1 + 1, \dots, x_s + 1> \exp(is\theta) \tag{3.8}$$

The Bethe ansatz form for the states is then given by

$$|k_1, \dots k_s> = \sum_\sigma A_\sigma \exp[\sum_i k_{\sigma(i)} x_i]|x_1, \dots x_s> \tag{3.9}$$

The translation operator acts on these states as

$$T|k_1, \dots k_s> = \exp(i(s\theta + \sum k_i))|k_1, \dots k_s> \tag{3.10}$$

Now, in order for a state to be translationally invariant we need that

$$s\theta + \sum k_i = 0 \mod (2\pi). \tag{3.11}$$

Notice that this condition is independent of the length of the spin chain, but it depends on the angle θ and the number of spins that are considered to be up.

Also notice that the Bethe ansatz state does not yet know about the boundary conditions, as we have not made the relations between x_0 and x_n explicit. The quasi-periodic boundary conditions on the chain are obtained by requiring that $|0, x_2, \dots, x_s> \equiv \exp(in\theta)|x_2, \dots, x_s, n>$. Implementing these conditions we obtain relations between A_σ and it's cyclic permutations. For example $A_{1,2,\dots,s} = e^{in\theta + ink_1} A_{2,\dots,s,1}$.

When considering integrability for the spin chain, the problem of finding the energies is simplified because the S matrix factorizes into products of 2-2 scattering. This matrix is elastic and takes the explicit form

$$S(k_1, k_2) = -\exp(ik_2 - ik_1)\frac{2\Delta - \exp(ik_1) - \exp(-ik_2)}{2\Delta - \exp(-ik_1) - \exp(ik_2)} \tag{3.12}$$

Let us call the associated S matrix $S(k_i, k_j)$ whenever we exchange them, so that $A_{213\ldots n} = S(k_1, k_2)A_{123\ldots n}$, etc, this relation can be turned into

$$\exp^{in(k_1+\theta)} S_{12}(k_1, k_2)\ldots S_{1s}(k_1, k_s) = 1. \tag{3.13}$$

A straightforward calculation shows that the energy of a spin wave with momentum k_1 is given by $E_1 = 1 + r^2 - 2r\cos k_1$, so the total energy of the spin wave configuration ends up being equal to

$$\sum E_i \tag{3.14}$$

For the particular case of q a root of unity $q = \exp(2\pi i t/k)$, with t, k coprime, we have $r = 1$ and we want to check that the spectrum of states with zero energy (BPS states) corresponds to the calculation done in [26].

A ground state should essentially consist of a state where all the k_i vanish, so that the energy is zero. The periodicity condition implies that $\exp(in\theta) = 1$, while the translation invariance implies that $s\theta = 0 \mod (2\pi)$. In particular this implies that the length of the chain needs to be a multiple of k, as well as the number of defects. It can be seen that so long as $n > s > 0$, this matches the result of the calculation of the chiral ring in [26]. There, it was found that there is exactly one chiral operator with those quantum numbers, although the ordering of the fields was not important when one studies the chiral ring as the cohomology of Q. This is because one is free to use the F-term equations of motion of the vacuum. This particular operator is in the untwisted sector of the orbifold theory.

The ground states $|\mathcal{G}\rangle$ for length b are also elements of the chiral ring. They are untwisted only if b is a multiple of k, otherwise they belong to the twisted sector.

We can also calculate the energy for two impurity states, in terms of solution of a transcendental equation. Let these have momenta k_1, k_2. The periodicity condition translates to

$$\exp(in(k_1 + \theta))S(k_1, k_2) = 1 \tag{3.15}$$

$$\exp(in(k_2 + \theta))S(k_2, k_1) = 1 \tag{3.16}$$

which adds up to $n(k_1 + k_2 + 2\theta)$ being a multiple of 2π. Also translation invariance amounts to $2\theta + k_1 + k_2$ also being a multiple of 2π. If we let $u = \exp(i(k_1 + \theta))$, then $u^{-1} = \exp(i(k_2 + \theta))$, and the Bethe equations above become

$$u^{n-2}\frac{2\Delta - 2u\cos\theta}{2\Delta - 2u^{-1}\cos\theta} = -1 \tag{3.17}$$

In the particular case $\Delta = 1, \theta = 0$ the system simplifies to give $(u^{n-1}-1)(1-u) = 0$. This is the value that corresponds to the $\mathcal{N} = 4$ SYM theory. It has been calculated in other papers [31, 32]. Here we use it as a consistency check. There are other special cases where the problem is easily solvable. Take for example $\cos(\theta) = 0$. In this case the polynomial becomes $u^{n-2} = -1$ independent of the value of Δ, this is the case of the $Z_4 \times Z_4$ orbifold. There are also simplifications if $\cos(\theta) = -1, \Delta = 1$, which correspond to the $Z_2 \times Z_2$ orbifold.

In any case, the equation (3.17) gives rise generically to a polynomial of degree n, which has n different roots. Notice also that if u is a root, then so is u^{-1}. We can use this Z_2 identification to simplify the polynomial further. This operation just corresponds to exchanging the momenta of the two spin waves, and gives rise to the same state. Thus for every pair of roots u, u^{-1} there is just one state associated to it. We can also write it as a polynomial for the variable $u + u^{-1}$ of lower degree.

Notice also that for u real $\cos\theta > 0$, $u \to +\infty$ the left hand side goes to $-\infty$, while at $u = \Delta/cos\theta$ the left hand side vanishes. The function for u real is real and continuous in that interval, so there is always a solution where u is real and greater than $2\Delta/cos(\theta)$. This signals a bound state of the two defects, because the relative momentum of the particles is complex (the wave function of the two particle system decays exponentially when we separate the particles). This is expected because the system is ferromagnetic. This property favours equal spins being next to each other, or for two spin waves to form a bound state. This state survives even for $\cos(\theta) = 0$, but it is missed by the polynomial equation, as $u \to \infty$. This is the only case where there is a real reduction of the rank of the polynomial because one of the roots goes to infinity.

The other simplifications occur from additional roots at $u = \pm 1$, which correspond to $\mathcal{N} = 4$ SYM or the $Z_2 \times Z_2$ orbifold.

There is one more thing that one should notice. In the spin chain with $\theta = 0$ the periodicity conditions imply that $k_1 + \cdots + k_s = 0 \mod 2\pi$. Given a solution of the Bethe equations with some collection of k_i, taking $k_i \to -k_i$ for all i produces (generically) a different state which also satisfies the Bethe equations and the periodicity condition, with the same energy as the original state. This operation behaves as parity on the worldsheet, and it corresponds to charge conjugation on the SYM side.

These two states with equal energy are called a parity pair [33], and their anomalous dimensions are related, even though they are not members of the same superconformal multiplet. This degeneracy was argued to be one of the hallmarks of integrability.

When we turn on the phase for q, we change the periodicity condition $\sum k_i = s\theta \mod 2\pi$, so taking $k_i \to -k_i$ produces a new state which would be degenerate with the original (as far as the sums of the one particle energies is concerned). However the state in question does not (generically) satisfy the periodicity condition! It follows

that the parity pairs of the $\mathcal{N}=4$ are generically lifted for a complex value of q.

4. The three state chain Hamiltonian

Following the study of the q deformed $\mathcal{N}=4$ theory, we can now consider the problem of diagonalizing the anomalous dimensions of chiral operators involving words formed with X, Y, Z. It is clear that we can make a three state spin chain with a basis for V_i given by three vectors $|0>, |1>, |2>$. We can also use elementary matrices $E_{ij}|k>=|i>\delta_{jk}$ to write the effective hamiltonian in the following form

$$H \sim (g^2Nh)\left[\sum q(E^i_{01}E^{i+1}_{10} + E^i_{12}E^{i+1}_{21} + E^i_{20}E^{i+1}_{02}) \right. \tag{4.1}$$

$$+q^*(E^i_{10}E^{i+1}_{01} + E^i_{21}E^{i+1}_{12} + E^i_{02}E^{i+1}_{20}) \tag{4.2}$$

$$+(E^i_{00}E^{i+1}_{11} + E^i_{11}E^{i+1}_{22} + E^i_{22}E^{i+1}_{00}) \tag{4.3}$$

$$\left. + q^*q(E^i_{11}E^{i+1}_{00} + E^i_{22}E^{i+1}_{11} + E^i_{00}E^{i+1}_{22})\right] \tag{4.4}$$

Again, it is trivial to show that H commutes with the operator that counts the number of X, Y, Z, and that it annihilates the Bethe-reference state

$$|\mathcal{G}\rangle = |0000\ldots 0> \tag{4.5}$$

Clearly, if we consider the subsystem where all of the vectors are any two of $|0>$ $, |1>, |2>$ we obtain an associated XXZ spin chain model, which reproduces all the results in the previous section, so the first guess one would make is that with all these subsectors being integrable, the spin chain above is integrable and it is analogous to the XXZ spin chain for $SU(3)$.

However, the story is not as simple. First we want to check that we can eliminate the phases from the spin chain Hamiltonian. The new ingredient we need to consider now is the presence of words with all three types of letters. Again, if $q = r\exp(i\theta)$ we can do a position dependent transformation on the vectors $|0>_k, |1>_k, |2>_k$ as follows

$$|0>_k = |\tilde{0}>_k;\ |1>_k = \exp(ik\theta)|\tilde{1}>_k;\ |2>_k = \exp(-ik\theta)|\tilde{2}>_k \tag{4.6}$$

This eliminates the phases of the terms $qE_{01}\otimes E_{10}$ and $q^*E_{02}\otimes E_{20}$ and their conjugates, but it triples the phase of the terms that involve $E_{12}\otimes E_{21}$. This is because the transformation between the bases satisfies

$$U\tilde{E}^k_{01}U^{-1} = e^{ik\theta}E^k_{01} \tag{4.7}$$

etc., so the change in the form of the Hamiltonian is solely due to the difference in phases between adjacent neighbors. In this new basis the spin chain becomes

$$\begin{aligned} H \sim (g^2Nh)\Big[\sum r(E^i_{01}E^{i+1}_{10} + E^i_{20}E^{i+1}_{02}) + r\exp(3i\theta)E^i_{12}E^{i+1}_{21}) \\ +r(E^i_{10}E^{i+1}_{01} + E^i_{02}E^{i+1}_{20}) + r\exp(-3i\theta)E^i_{21}E^{i+1}_{12} \\ +(E^i_{00}E^{i+1}_{11} + E^i_{11}E^{i+1}_{22} + E^i_{22}E^{i+1}_{00}) \\ + r^2(E^i_{11}E^{i+1}_{00} + E^i_{22}E^{i+1}_{11} + E^i_{00}E^{i+1}_{22})\Big] \end{aligned} \tag{4.8}$$

In this basis we have just described the boundary condition is still simple enough, but we have this extra phase in the Hamiltonian that we have not eliminated. We will now calculate a change of basis that is more complicated, but where we can eliminate the phase of q completely. The advantage is that the model looks a lot more symmetric in this newer basis, and identical to the $\mathcal{N}=4$ spin chain if q is unitary. All the information of the theory as a function of the phase of q gets then pushed to the boundary conditions.

The new change of basis requires adding extra phases which involve the order of the vectors $|2>, |1>$ independent of how many $|0>$ they have between them. The reader can convince themselves that this is possible, so that q can be taken to be real. The phase we need is exactly $\exp^{3i\theta}$ whenever a vector $|\tilde{2}>$ passes from being on the left of a vector $|\tilde{1}>$ to the right of it, which is the opposite of the shift done from passing a vector $|\tilde{2}>$ to the right of a vector $|\tilde{0}>$. This transformation is obtained (on an open chain) by $U|\psi_{New}>=|\psi>$, where

$$U = \exp(i\theta g(\psi)) \tag{4.9}$$

whenever ψ is one of the basis vectors of the Hilbert space of states. This function $g(\psi)$ depends only on the ordering of the spins in ψ that take the value $|1>$ and $|2>$. Let us call this order Λ. The function $g(\Lambda)$ counts how many 1 appear before states 2 with multiplicity. For example $g(112) = 2$, while $g(1122) = 4$ and $g(1212) = 3$. This factor takes into account the different orderings of the fields after eliminating the phases. It also vanishes if all the letters are alike.

In the position basis the translation operator now acts by adding 1 to each of the x_i and adding the phase $\exp(i(N_1 - N_2)\theta)$ which depends on the numbers of vectors which are set to be equal to $|1>$ and $|2>$.

Similarly, the periodicity conditions become more complicated because we need to keep track of the number of vectors set to $|1>$ and $|2>$. This periodicity condition will depend on an ordering of the $|1>, |2>$ vectors. Then if we introduce the position basis with order Λ we get

$$|0, x_1, \ldots, x_s>^{\Lambda} = |x_1, \ldots, x_s, n>^{C(\Lambda)} \exp(in\theta f(\Lambda)) \exp(+3i(g(\Lambda) - g(C(\Lambda))\theta) \tag{4.10}$$

where C changes the ordering given by Λ cyclically, and $f(\Lambda) = 1$ if the ordering Λ ends in $|1>$, while it is given by -1 if the order ends in $|2>$.

This change of basis shows that the difference between the second basis and the third is an additional phase shift depending on the ordering of the particles in the state. This amounts to an additional phase shift in the S-matrix when a particle of type 2 jumps to the right of a particle of type 1.

Again, if we look at the case $r = 1$ and q a root of unity (namely $k\theta = 0 \mod (2\pi)$), all the effect of q is to give twisted boundary conditions. In this case we know that the spin chain model is integrable and can be solved using Bethe

ansatz. If we look for ground states, then all the momenta of the excitations should vanish with all $k_i \sim 0$, the translation invariance condition implies $N_1\theta - N_2\theta = 0 \mod (2\pi)$, which sets $N_1 - N_2$ equal to a multiple of k. The periodicity condition then imposes $3N_1\theta - n\theta = 0 \mod (2\pi)$. We can rewrite this in the following form $n = N_0 + N_1 + N_2 = 3N_1 \mod (k)$, but $N_2 = N_1 \mod (k)$, so the condition boils down to $N_0 = N_1 = N_2 \mod (k)$. This is exactly the result expected from the computation of the elements of the chiral ring in [25, 26].

Now let us study the case with $r \neq 1$. We want to ask whether the spin chain above is integrable or not. It turns out that it is not integrable. We will show that the associated system, although it should be solvable by Bethe ansatz, the associated S-matrix fails to satisfy the Yang-Baxter equation, so the Bethe ansatz is inconsistent. Thus, this deformation is non-integrable, contrary to the conjectures phrased in [37].

To do the calculation explicitly, we need to consider scattering of an excitation of momentum k_1 (staring to the left) with an excitation of momentum k_2, such that they have different spin labels. This is done with respect to the reference state $|\mathcal{G}\rangle$, so we have one defect $|1>$ and one defect $|2>$, and we have two possible initial states and two possible final states.

Since the interactions respect the spin labels, in the final state we get two particles of momenta k_2 and k_1, but they might have exchanged the spin labels. The end result is a 2×2 matrix, where the two initial and final states differ by the labels of the spins. This is part of a 4×4 S matrix with the initial and final states given by the two possible spin labels of each particle. This 2×2 S-matrix for different spins S^{ij}_{kl} is given by

$$S(k_1, k_2) = \begin{pmatrix} S^{12}_{12} & S^{21}_{12} \\ S^{12}_{21} & S^{21}_{21} \end{pmatrix} = -M(q_1, q_2)M^{-1}(q_2, q_1) \tag{4.11}$$

where $q_i = \exp(ik_i)$, and

$$M(q_1, q_2) = \begin{pmatrix} 1 - 2r^2 + rq_1 + rq_2^{-1} & -r \\ -r & r^2 - 2 + rq_1 + rq_2^{-1} \end{pmatrix} \tag{4.12}$$

Once this S-matrix has been calculated, we can build the 4×4 S-matrix with the scattering phases from the spin 1/2 XXZ model. Consistency of the Bethe ansatz for more then two particles impurities, together with factorization implies that the S-matrix of scattering defects satisfies the Yang-Baxter equation (this is standard material in integrable spin chains. The reader unfamiliar with these facts is encouraged to read some review articles and books which we have found useful [34, 35, 36]). With the specific form of the S-matrix given above , supplemented by the S matrix of the XXZ spin chain to obtain the full 4×4 S-matrix it can be verified numerically that the Yang Baxter equation does not hold, but that in the special case $r = 1$ it does hold. This is consistent with our discussion so far as we have argued that in the case $r = 1$ we get the same spin chain as $\mathcal{N} = 4$ SYM theory.

We can trace the lack of integrability to the fact that the matrix M above is too complicated. Another choice of M which is simpler, and leads to an S-matrix which does satisfies the Yang-Baxter relation is given by

$$M(q_1, q_2) = \begin{pmatrix} -r^2 + rq_1 + rq_2^{-1} & -r \\ -r & -1 + rq_1 + rq_2^{-1} \end{pmatrix} \tag{4.13}$$

or equivalently, if we multiply M by r^{-1} (which will not affect the form of S),

$$M(q_1, q_2) = \begin{pmatrix} -r + q_1 + q_2^{-1} & -1 \\ -1 & -r^{-1} + q_1 + q_2^{-1} \end{pmatrix} \tag{4.14}$$

Clearly, this choice corresponds to a different spin chain Hamiltonian, which is not obtained from an $\mathcal{N} = 1$ deformation of the $\mathcal{N} = 4$ SYM spin chain. We will return to this model later in the paper in section 5

Now, let us for a while concentrate on understanding how the spectrum of orbifolds is related to the theory which is not orbifolded. The basic issue is to see that in the untwisted sector, the equations which describe the energy of the states are the same.

The main point is to understand how the periodicity condition works, as we have already seen that the bulk of the spin chain model is the same. Let us assume that $m\theta = 0 \mod (2\pi)$, this is $q^m = 1$. The untwisted sector of the theory is then characterized by requiring [25]

$$N_{|1>} - N_{|2>} = N_{|2>} - N_{|0>} = 0 \mod m \tag{4.15}$$

The periodicity condition has a phase which is given by

$$in\theta f(\Lambda)) + 3i(g(\Lambda) - g(C(\Lambda))\theta, \tag{4.16}$$

with $n = N_{|0>} + N_{|1>} + N_{|2>}$. If Λ ends in $|2>$, then $g(\Lambda) - g(C\Lambda) = -N_{|1>}$. The total phase is then

$$\theta(N_{|2>} - N_{|1>}) + \theta(N_{|0>} - N_{|1>}) = 0 \mod (2\pi) \tag{4.17}$$

since both $N_{|2>} - N_{|1>}$ and $N_{|0>} - N_{|1>}$ are multiples of m. The equation can also be checked in the case that we end Λ with $|1>$. This means that the boundary condition for this sector is the same as for the theory where we have not included the phase θ at all. Therefore for these states there are no corrections to the Bethe equations that depend on θ, and the system is solved by the same states that the original $\mathcal{N} = 4$ theory is solved by.

Other states of the orbifold theory will receive corrections that depend on θ, and these are the states which belong to the twisted sector of the theory. These equations will be more complicated than Eq.(3.17), because the scattering matrices are honest matrices and not just a collection of phases: the S-matrices include all the possible orderings of the defects on the chain.

5. Towards non-supersymmetric integrable deformations

As we have seen in the previous section, superconformal deformations of $\mathcal{N} = 4$ SYM which preserve a $U(1)^3$ symmetry are generically non-integrable. There are some special cases which are integrable, particularly we require that $|q| = 1$ and a dense set of them corresponds to taking orbifolds of $\mathcal{N} = 4$ SYM, these are the ones with $q^n = 1$ for some integer n. These have the property that after a position dependent change of variables, the spin chain model is identical to the one of $\mathcal{N} = 4$, but the boundary conditions are twisted.

We can now try to consider deformations of the field theory which change the microscopic spin chain model such that they do not correspond to a twisting of the original spin chain, but that still preserve conformal invariance. These can not be supersymmetric. We also mentioned in passing some construction of the scattering amplitude as a function of r (via Eq. (4.13)) which did give rise to integrability. It is easy to write the associated spin chain model (assume $q = q^* = r$ is real for clarity) as follows

$$\begin{aligned} H \sim (g^2 Nh) \Big[\sum r(E^i_{01}E^{i+1}_{10} + E^i_{12}E^{i+1}_{21} + E^i_{20}E^{i+1}_{02}) \\ +r(E^i_{10}E^{i+1}_{01} + E^i_{21}E^{i+1}_{12} + E^i_{02}E^{i+1}_{20}) \\ +(E^i_{00}E^{i+1}_{11} + E^i_{11}E^{i+1}_{22} + \underline{E^i_{00}E^{i+1}_{22}}) \\ + r^2(E^i_{11}E^{i+1}_{00} + E^i_{22}E^{i+1}_{11} + \underline{E^i_{22}E^{i+1}_{00}})\Big] \end{aligned} \tag{5.1}$$

The only difference in the spin chain model is that we have exchanged the coefficients of the two terms which are underlined above.

It is easy to see that this would be the effect of an "F-term" Lagrangian of the following form

$$\begin{aligned} \mathrm{tr}((XY - rYX)(\bar{Y}\bar{X} - r\bar{X}\bar{Y})) + \mathrm{tr}((YZ - rZY)(\bar{Z}\bar{Y} - r\bar{Y}\bar{Z})) \\ +\underline{\mathrm{tr}((XZ - rZX)(\bar{Z}\bar{X} - r\bar{X}\bar{Z})} \end{aligned} \tag{5.2}$$

where the order of the r-deformed commutators squared has been changed in the underlined term in the Lagrangian, and, for the sake of argument, let us keep the D-terms of the Lagrangian fixed so that the same cancellations between D-terms and photon exchange that are available for $r = 1$ are still possible when $r \neq 1$ at one loop. Notice that these are flavor blind with respect to spin chains with only X, Y, Z letters on them, so they can in principle modify the above result by adding a term proportional to the identity. This is a trivial operation at the level of this sector and does not spoil the partial integrability. Clearly this would have to be studied in more detail to ensure that the full list of single trace operators leads to an integrable model (partial results were found in [37]).

This deformed theory has been obtained from $\mathcal{N} = 4$ by a deformation of the potential which looks marginal at the free field theory level. The deformation preserves the $U(1)^3$ symmetry of the original theory, but it breaks the Z_3 symmetry $X \to Y \to Z \to X$. Moreover, we have a set of operators which are remnants of chiral operators for a choice of $\mathcal{N} = 1$ splitting of the original fields of $\mathcal{N} = 4$ SYM, which are made of X, Y, Z alone, and their charges are such that they don't mix with other operators.

At the level of the spin chains, it is interesting to ask what kind of symmetries the spin chain coming from this potential has. It turns out that the spin chain as built above, just like the XXZ model is a deformation of the XXX model with $SU(2)$ symmetry, is a member of generalizations of the XXZ model which are available for any system with Lie group symmetry. The symmetries associated to these deformations are given by corresponding quantum groups.

The spin chain described above has $SU_q(3)$ symmetry, and at each site (reiterating that we are only considering words made out of X, Y, Z) we have a 3 dimensional representation of $SU_q(3)$. One can generalize very easily the discussion to having spin chains for $SU_q(M)$, with a spin degree of freedom in the fundamental and nearest neighbor interactions, so for the time being we will discuss this more general case.

We are interested in computing the Hamiltonian of such a spin chain which is integrable. Before we do that however, we need to make a small excursion into quantum groups, to describe what it means for a spin chain Hamiltonian to be invariant under a quantum group symmetry. The literature on quantum groups is very extensive. Here we point out the following two books as a place to begin reading about them[34, 40] which will be more appealing to physicists. Here we give a brief list of properties of the quantum group deformations of Lie algebras which we will use.

5.1 A brief introduction to the quantum groups $SU_q(M)$

The basic idea of quantum groups as used in this paper is that they are (one-parameter) deformations of universal enveloping algebras of Lie algebras, which permits us to have most of the properties of the representation theory of the Lie algebra carry through to the quantum group version of it.

These algebras for our purposes are algebras over the complex numbers, they are associative, they have an identity, and they can be written in terms of a set of generators with some relations. The defining relations are best written in terms of the Chevalley basis for the Lie algebra. We take a set of positive and negative roots E_α, F_α associated to each element of the Cartan algebra.

We will denote the quantum group algebra by $\mathcal{A}$.

The defining relations are given by

$$
\begin{aligned}
[H_\alpha, H_\beta] &= 0 && (5.3)\\
[H_\alpha, E_\beta] &= C_{\alpha\beta}E_\beta && (5.4)\\
[H_\alpha, F_\beta] &= -C_{\alpha\beta}F_\beta && (5.5)\\
[E_\alpha, F_\beta] &= \delta_{\alpha\beta}\frac{q^{H_\alpha}-q^{-H_\alpha}}{q-q^{-1}} && (5.6)
\end{aligned}
$$

In the above notation $C_{\alpha\beta}$ is the Cartan matrix of the associated Lie algebra, and q is a complex number.

There are additional relations called Serre relations which will be automatically satisfied for all representations we construct, so we can ignore them for our present discussion.

When we take the limit $q \to 1$ these relations turn exactly to the Lie algebra relations of the corresponding generators in the Lie algebra.

Representations of the above algebra in terms of $k \times k$ matrices furnish a k-dimensional representation of the quantum group.

For $SU_q(M)$, there are $M-1$ generators of the Cartan, and the fundamental representation is characterized by M vectors e_i, from $i = 1, \ldots, M$, where

$$
\begin{aligned}
H_\alpha e_i &= \delta_{\alpha,i}e_i - \delta_{\alpha+1,i}e_i && (5.7)\\
F_\alpha e_i &= \delta_{\alpha,i}e_{i+1} && (5.8)\\
E_\alpha e_i &= \delta_{\alpha,i-1}e_{i-1} && (5.9)
\end{aligned}
$$

One property which quantum groups share with the Lie algebras themselves is that it is possible to tensor multiply quantum group representations and obtain new representations of the quantum group (this is like addition of angular momenta). Clearly, a tensor product of two representations $R_1 \otimes R_2$ is a representation of the tensor product $\mathcal{A} \otimes \mathcal{A}$, but there is an algebra homomorphism

$$
\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \tag{5.10}
$$

called the coproduct, which turns representations of $\mathcal{A} \otimes \mathcal{A}$ into representations of $\mathcal{A}$ and which acts on the generators as follows

$$
\begin{aligned}
\Delta(H_\alpha) &= H_\alpha \otimes 1 + 1 \otimes H_\alpha && (5.11)\\
\Delta(E_\alpha) &= E_\alpha \otimes 1 + q^{H_\alpha} \otimes E_\alpha && (5.12)\\
\Delta(F_\alpha) &= F_\alpha \otimes q^{-H_\alpha} + 1 \otimes F_\alpha && (5.13)
\end{aligned}
$$

The coproduct is associative, meaning that given three representations R_i, the tensor products $(R_1 \otimes R_2) \otimes R_3$ is canonically identified as a representation with $R_1 \otimes (R_2 \otimes R_3)$, so that we can drop parenthesis on the tensor products.

Starting from the fundamental representation, it is possible to construct all finite dimensional representations of $SU_q(M)$ by taking tensor products, so long as q is generic. Indeed, the representations behave under tensor product essentially the same way as the representations of $SU(M)$. Any finite dimensional representation of the quantum group can be split, for generic q, as a direct sum of irreducibles. Also, the representations can be characterized by the same Young Tableaux and branching rules, however the Clebsch-Gordon coefficients relating the bases of the representations are deformed by q.

There is also a trivial representation where H, E, F all act as zero, and 1 is mapped to 1. This is a map from $\mathcal{A} \to \mathbb{C}$ which is called the coidentity. Tensoring with this representation acts like the identity with respect to the tensor product. Some of the axioms of quantum groups reflect these properties.

Finally, there is also a map which lets us turn right modules of the algebra into left modules and viceversa. This is the antipode map.

The main difference with Lie algebras is that canonical identifications which hold between modules of $SU(M)$ like $V_1 \otimes V_2 \sim V_2 \otimes V_1$ are not canonical any more. This is expressed by saying that the coproduct is not co-commutative.

The interested reader should consult the very extensive literature on quantum groups for a more detailed exposition of the properties of quantum groups.

5.2 Spin chains with quantum group symmetry

Now that we have the basic setup for groups $SU_q(M)$, we can consider an open spin chain whose Hilbert space is a tensor product of representations of $SU_q(M)$, $V_1 \otimes V_2 \otimes \cdots \otimes V_t$. The spin chain is called homogeneous if all the representations V_i are the same.

We say a Hamiltonian H acting on this vector space is nearest neighbor if

$$H \sim \sum H_{i,i+1} \tag{5.14}$$

where $H_{i,i+1}$ only acts on the subfactor $V_i \otimes V_{i+1}$, and the Hamiltonian is homogeneous if the spin chain is homogeneous and the Hamiltonian is translation invariant $H_{i,i+1} \sim H_{i+1,i+2}$. Since each element of the vector space belongs to a tensor product of representations of the quantum group, there is a global quantum group action on the whole Hilbert space by taking successive coproducts of the quantum group generators [1]. A Hamiltonian is said to respect the quantum group symmetry if the action of H on the Hilbert space commutes with the generators of the global quantum group.

For nearest neighbor interactions this becomes simple to analyze. A straightforward calculation shows that the problem can be reduced to how the quantum group acts on a nearest neighbor pair only $V_i \otimes V_{i+1}$.

[1] This action is unique because of the associativity of the coproduct.

Starting with the fundamental representation, the tensor product $V \otimes V = S \oplus A$ is split into just two irreducibles the q-symmetric and q-antisymmetric representations.

It can be shown that for generic q there are no non-trivial module maps between them. A quick proof can be obtained by showing that the Casimirs for both representations are different.

A map from $H : V \otimes V \to V \otimes V$ will respect the quantum group symmetry if it is a module map with respect to the algebra. Under the decomposition into irreducibles this has to be determined by projectors into the different irreducibles

$$H = aP_S + bP_A \tag{5.15}$$

That these are the only matrices which give the desired result follows from irreducibility of the representations, and the lack of module maps between different representations.

This can also be written as

$$H = aId + (b-a)P_A \tag{5.16}$$

a sum of the identity operator (which trivially preserves the quantum group symmetry) and the projection into the antisymmetric tensor representation.

We want to calculate this last projector explicitly.

All we need to do is calculate the antisymmetric tensor representation of $SU_q(M)$. This can be done by using a highest weight construction of the tensor representation. By definition a highest weight state will be the one that is annihilated by all the raising operators (these are the E_α's). In the fundamental representation the highest weight state will be the state we called e_1. The highest weight state of the tensor product $V \otimes V$ is $e_1 \otimes e_1$, but this is a member of the symmetric representation. Acting with one lowering operator F_1 we get the state $e^S_{12} = e_1 \otimes e_2 + q^{-1} e_2 \otimes e_1$. There is another state with the same weights under the Cartan, which is of the form $e^A_{12} = e_1 \otimes e_2 + B e_2 \otimes e_1$. This state is automatically annihilated by $E_2, \dots E_{M_1}$. Requiring that this state be annihilated by E_1 fixes the value of B to be equal to $-q$. Notice that if we require the H_α to be Hermitian operators, then the states $e_1, \dots e_n$ are orthogonal, and the states e^S_{12} and e^A_{12} will be orthogonal with respect to the natural norm of the product if q is real.

Having the highest weight e^A_{12}, we can act with the lowering operator F_2, and then we obtain the state $e_1 \otimes e_3 - q e_3 \otimes e_1$. Notice that there are no additional factors of q because e_1 is neutral with respect to H_2. Similarly we can keep on acting with lowering operators and obtain the state $e_2 \otimes e_3 - q e_3 \otimes e_2$. A straightforward computation shows that the basis of the q-antisymmetric tensor representation is given by all the vectors

$$e^A_{ij} = e_i \otimes e_j - q e_j \otimes e_i, \; i < j \tag{5.17}$$

The projector of the tensor product into this representation will be given (up to a normalization factor t) by

$$P_A = t\sum_{i<j}(e_i \otimes e_j - qe_j \otimes e_i)(\hat{e}_i \otimes \hat{e}_j - q\hat{e}_j \otimes \hat{e}_i) \tag{5.18}$$

where the $\hat{e}_i$ are a dual basis for the e_i. [2] The dual basis satisfies $\hat{e}_j.e_i = \delta_{ij}$. t can also be calculated readily to be equal to $1/(1-q^2)$.

It is easy to verify that $(\hat{e}_1 \otimes \hat{e}_2 - q\hat{e}_2 \otimes \hat{e}_1) \cdot e^S_{12} = 0$ and this generalizes to all members of the symmetric representation, so that the above construction is indeed a projector. We can write these in terms of the elementary matrices acting on each component of the tensor product $E_{ij} = e_i\hat{e}_j$ as follows

$$P_A = t\sum_{i<j} E_{ii} \otimes E_{jj} + qE_{ij} \otimes E_{ji} + qE_{ji} \otimes E_{ij} + q^2E_{jj} \otimes E_{ii} \tag{5.19}$$

Compare this expression to equation (5.2) and we find perfect agreement.

The fact that this spin chain is integrable follows from general constructions of integrable systems based on universal R-matrices, basically it states that given any representation of a quantum group as described above, there is an associated spin chain model which is integrable (and of nearest neighbor type, see for example [34]). We shall return to this discussion in the next section. In this case, the part of the Hamiltonian that is proportional to the identity is trivial, as adding the identity to any diagonalizable Hamiltonian produces a new Hamiltonian which is diagonalizable in the same basis.

The fact that the Hamiltonian is non-trivial and respects the quantum group invariance is enough to make it integrable, so long as we are dealing with the fundamental representation of $SU_q(N)$. In light of this fact, the fact that [19] found integrability in an $SU(3)$ subsector of a theory, where the spin chain was in the fundamental of $SU(3)$ is not surprising at all: it could not have been otherwise.

For higher representations of $SU_q(M)$ the integrability is not immediate as there are more than two representations appearing in the tensor product $V \otimes V$. Even after we remove the identity, there are at least two complex numbers that need to be tuned just right for the system to be integrable. Formal constructions of these models can be obtained from looking at derivatives of trigonometric R-matrices for $SU_q(N)$, see for example [41]. In practice, it is very hard to calculate them in explicit form.

6. $SO(6)$ integrable spin chains

So far we found that generic supersymmetry preserving deformations of the maximally supersymmetric Yang-Mills destroy the delicate integrability of the matrix

[2] Remember that for any algebra the dual module to a given right module can be made into a left module.

of conformal dimensions. It is encouraging, however, to find a nonsupersymmetric deformation that preserves the integrability of the chiral sector. In this section we seek a nonsupersymmetric deformation that would preserve integrability at one loop level among all single-trace operators involving only scalars.

In the language of spin chains, in the case of $N = 4$ super-Yang-Mills, chiral operators built out of X and Y:

$$\mathrm{tr}(XXYXYY \dots XYYY), \tag{6.1}$$

correspond to an $SU(2)$ XXX spin chain, while the chiral operators involving all scalars

$$\mathrm{tr}(XZZYX \dots YYXZ), \tag{6.2}$$

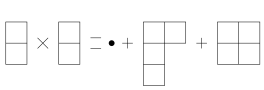
correspond to an $SU(3)$ XXX spin chain. Finally, the operators built out of scalars $X, Y, Z, \bar{X}, \bar{Y}$, and $\bar{Z}$:

$$\mathrm{tr}(X\bar{Y}Y\bar{Z}\bar{Z}Y \dots \bar{X}Z) \tag{6.3}$$

correspond to an $SO(6)$ XXX spin chain [6]. It is natural to expect that any deformation of the theory that would respect integrability would be a deformation of these spin chains. As described in section 3 any supersymmetry preserving deformation of Yang-Mills results in deforming the XXX $SU(2)$ spin chain into the XXZ $SU(2)$ spin chain. However, for the larger $SU(3)$ sector, we find that this deformation of Yang-Mills contorts the XXX $SU(3)$ spin chain into a system that is no longer integrable. In its turn the $SU(3)$ system does have an integrable deformation that can be called XXZ. Indeed, it corresponds to a deformation of the $N = 4$ Yang-Mills that breaks all supersymmetry. As mentioned above this XXZ spin chain has quantum group of symmetries that can be thought of as a deformation of the corresponding part of R-symmetry. With this in mind one might wonder whether it is this deformed symmetry that is responsible for integrability. Thus here we look for a deformation of the XXX $SO(6)$ integrable spin chain that is $SO_q(6)$ symmetric, we shall refer to it as an $SO(6)$ XXZ spin chain. Given the corresponding spin chain Hamiltonian one can search for the Yang-Mills theory Lagrangian that would give rise to it.

6.1 Symmetry considerations

Our purpose here is to find a general form of a nearest-neighbor Hamiltonian of a homogeneous spin chain with $SO_q(6) \sim SU_q(4)$ symmetry. We proceed as in subsection 5.2. Since the tensor product of two fundamental representations of $SO(6)$ decomposes into a singlet, a 15- and a 20-dimensional representation



6 6 1 15 20

the Hamiltonian has to be of the form

$$H = a\mathrm{Id} + b' P_{\mathrm{singlet}} + c' P_{15} = a\mathrm{Id} + bSS^\dagger + c\sum_{j=1}^{15} w_j w_j^\dagger. \tag{6.4}$$

The difference between $b.b'$ and c, c' is that we choose to write the projectors in a non-normalized fashion on the right hand side. The singlet can easily be identified

$$S = q^{-2}X\bar{X} + q^2\bar{X}X - q^{-1}Y\bar{Y} - q\bar{Y}Y + Z\bar{Z} + \bar{Z}Z. \tag{6.5}$$

The relevant details of $SO_q(6)$ can be found in the appendix. For our purposes it suffices to present the zero-weight orthonormal elements of the 15-dimensional representation

$$\begin{aligned}
w_7 &= \frac{Z\bar{Z} - \bar{Z}Z}{\sqrt{2}},\\
w_8 &= \frac{X\bar{X} - \bar{X}X + q^{-1}Y\bar{Y} - q\bar{Y}Y}{q+q^{-1}},\\
w_9 &= \sqrt{\frac{2}{q^2+q^{-2}}}\frac{X\bar{X} - \bar{X}X - qY\bar{Y} + q^{-1}\bar{Y}Y + (q^2-q^{-2})(Z\bar{Z}+\bar{Z}Z)/2}{(q+q^{-1})}.
\end{aligned} \tag{6.6}$$

6.2 Exact $SO(6)$ XXZ Hamiltonian

In the previous subsection we have obtained the general form of a Hamiltonian with $SO_q(6)$ symmetry. Here, however, we obtain the exact Hamiltonian for the $SO(6)$ XXZ integrable spin chain. In other words we find the relation between the coefficients of the above Hamiltonian so that it is integrable. To achieve this we use a modification of the method for producing the R-matrices for higher representations that can be found in [42], as well as the following construction of [43] to produce the Hamiltonian.

Given group G and its representation V one constructs an R-matrix which depends on two parameters u and q. For each value of u the R-matrix describes a linear map $R(u) : V \otimes V \to V \otimes V$, which satisfies the Yang-Baxter equation

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v), \tag{6.7}$$

where $R_{12}(u) = R(u)\otimes I$, etc. Given an R-matrix one constructs the nearest neighbor interaction Hamiltonian

$$H = \sum_{1}^{k} H_{i,i+1}, \tag{6.8}$$

with $H_{i,i+1}$ acting on the tensor product of the representations at the i-th and $(i+1)$-st site $V_i \otimes V_{i+1}$ and it is defined using the R-matrix as

$$H_{i,i+1} = \left(\frac{d}{du}R_i(u)|_{u=0}\right) R_i(0)^{-1}. \tag{6.9}$$

The integrability of this system can be inferred from the following property of the transfer matrix

$$t(u) = \mathrm{tr}_{\text{second indices}} R_1(u) R_2(u) \dots R_n(u). \tag{6.10}$$

What is meant here is that we can consider each R-matrix R_j as a linear map from $V_j \otimes U$ to itself, with the same space U at each point. The trace in the above expression is over the U space only.

Then it follows that the coefficients J_l of the Taylor expansion

$$\ln t^{-1}(0)t(u) = J_1 u + J_2 \frac{u^2}{2} + \dots + J_k \frac{u^k}{k!} + \dots \tag{6.11}$$

all commute and J_l involves only l-neighbour interactions. Moreover $J_1 = H$. Thus the above Hamiltonian is integrable. It can also be verified that it is invariant with respect to the quantum group $U_q(G)$.

With this in mind, let us look for the trigonometric $SO(6)$ R-matrix, so that it depends nontrivially on the parameter u, as well as on the q-deformation parameter $q = \exp(-\alpha)$.

There is extensive literature containing various solutions of the Yang-Baxter equation. For example [34] contains a list of R-matrices for all classical Lie groups. One can find a universal R-matrix in e.g. [40]. In these cases, however, the R-matrix contains only q-dependence and in independent of the parameter u. The above method would produce a trivial Hamiltonian when applied to these matrices.

In [44] Reshetikhin presented $O(n)$ and $Sp(2n)$ invariant R-matrices. These, however, have rational u dependence and no dependence on q parameter, and therefore lead to the corresponding XXX spin chains via the method described above.

For our purposes it suffices to know the R-matrix for $SO(6)$ group only, however, it is crucial that it has nontrivial dependence on both q and u parameters. Once again, we think of the fundamental of $SO(6)$ as an antisymmetric representation of $SU(4)$. Now, having an explicit R-matrix for $SU(4)$ with q and u dependence, we use the following idea of [42] to obtain an R-matrix for $SO(6)$. The trigonometric R-matrix for $SU(n)$ can be found in [45] and has the following nonzero entries

$$\begin{aligned} R_{jj,jj} &= \sinh(u+\alpha) \\ R_{jk,jk} &= \sinh(u); \qquad\qquad j \neq k \\ R_{jk,kj} &= \sinh(\alpha) e^{-u\,\mathrm{sign}[j-k]}; \;\; j \neq k. \end{aligned} \tag{6.12}$$

We recall that $q = e^{-\alpha}$.

It is crucial to note that for $u = -\alpha$ this R-matrix is proportional to the projector $R(-\alpha) = (-2\sinh\alpha) P_q^-$. Where P_q^- is the q-projector. Since there is an ambiguity in defining the q-antisymmetrization we note that $\left(P_q^-\right)_{jk,kj} = -q^{\mathrm{sign}(k-j)}/2$, for $j \neq k$. It follows from the Yang-Baxter equation that the 'pair-to-one' R-matrix

$$R_{12,3}(u) = R_{13}(u - \alpha/2) R_{23}(u + \alpha/2) \tag{6.13}$$

acts on $V \otimes V \otimes V$ and satisfies so called triangularity relation

$$P^{-}_{12} R_{12,3} P^{+}_{12} = 0. \tag{6.14}$$

This relation implies that it is consistent to restrict to the q-antisymmetric representation in the first two of the tree representations. The 'pair-to-pair' R-matrix

$$R_{12,34}(u) = R_{23}(u-\alpha) R_{24}(u) R_{13}(u) R_{14}(u+\alpha), \tag{6.15}$$

evidently satisfies the Yang-Baxter equation by virtue of Eq.(6.7). It also satisfies the triangularity condition[3]

$$\begin{aligned} P^{-}_{21} R_{12,34}(u) P^{+}_{21} &= 0, \\ P^{-}_{43} R_{12,34}(u) P^{+}_{43} &= 0. \end{aligned} \tag{6.16}$$

It follows that its q-antisymmetrization

$$R_{[12],[34]}(u) = P^{-}_{21} P^{-}_{43} R_{12,34} P^{-}_{43} P^{-}_{21} \tag{6.17}$$

satisfies the Yang-Baxter relation as well. This R-matrix acts on the antisymmetric representations of $SU(n)$ only and in the case of $n = 4$ is exactly the R-matrix for the fundamental of $SO(6)$ that we need.

The result of this computation is the following R-matrix

$$\begin{aligned} R(u) = -\frac{\sinh(u-\alpha)}{(\sinh(u+\alpha))^2 \sinh(u+2\alpha)} \Bigg\{ & \sinh(u+\alpha)\sinh(u+2\alpha) \sum_{i=1}^{6} E_{ii} \otimes E_{ii} + \\ & + \sinh u \sinh(u+\alpha) \sum_{i=1}^{6} E_{ii} \otimes E_{\bar{i}\bar{i}} + \sinh u \sinh(u+2\alpha) \sum_{i \neq j, \bar{j}} E_{ii} \otimes E_{jj} + \\ & + \sinh\alpha \sinh(u+2\alpha) \sum_{i \neq j, \bar{j}} e^{-u \operatorname{sign}(i-j)} E_{ij} \otimes E_{ji} + \\ & + 2(\sinh\alpha)^2 \left[\cosh\alpha (e^{2u} E_{1\bar{1}} \otimes E_{\bar{1}1} + e^{-2u} E_{\bar{1}1} \otimes E_{1\bar{1}}) + \right. \\ & + \cosh(u+\alpha)(e^{u} E_{2\bar{2}} \otimes E_{\bar{2}2} + e^{-u} E_{\bar{2}2} \otimes E_{2\bar{2}}) + \\ & \left. + \cosh\alpha (E_{3\bar{3}} \otimes E_{\bar{3}3} + E_{\bar{3}3} \otimes E_{3\bar{3}}) \right] + \\ & - \sinh\alpha \sinh u \sum_{i \neq j, \bar{j}} \operatorname{sign}(i-j) e^{-(u+2\alpha)\operatorname{sign}(i-j)} (-e^{\alpha})^{\hat{i}-\hat{j}} E_{ij} \otimes E_{\bar{i}\bar{j}} \Bigg\} \end{aligned}$$

where

$$\begin{aligned} \bar{j} &= 7 - j, \\ \hat{j} &= \begin{cases} j + 1/2, & j \leq 3 \\ j - 1/2, & j > 3. \end{cases} \end{aligned}$$

[3]Notice the reversed order of indices in the q-projectors. Projectors have to be transposed in order to take advantage of the Yang-Baxter equation to verify triangularity.

This leads to the following Hamiltonian[4]

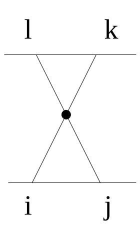
$$
\begin{aligned}
H &= \frac{q^2 - q^{-2}}{2} \left[\frac{d}{du} R(u) \right] \Bigg|_{u=0} R(0)^{-1} = \\
&= (q+q^{-1})^2 \sum E_{ii} \otimes E_{jj} + (q-q^{-1})^2 \sum E_{i\bar{i}} \otimes E_{\bar{i}i} - (q+q^{-1}) \sum E_{ij} \otimes E_{ji} + \\
&\quad +(q+q^{-1}) \sum \underline{q^{\mathrm{sign}(i-j)}} E_{ii} \otimes E_{jj} + \sum \underline{(-q)^{\hat{i}-\hat{j}} q^{2\mathrm{sign}(j-i)}} E_{i\bar{j}} \otimes E_{\bar{i}j}.
\end{aligned}
\tag{6.18}
$$

In the previous subsection we used only symmetry considerations to obtain the general form of the Hamiltonian. This Hamiltonian, indeed, has the form of Eq.(6.4) with the following values of the parameters

$$
\begin{aligned}
a &= \left(q + q^{-1}\right)^2, \\
b &= \frac{2}{q^2 + q^{-2}}, \\
c &= \left(q + q^{-1}\right)^2.
\end{aligned}
\tag{6.19}
$$

6.3 Corresponding Lagrangian deformation

Having the exact form of the integrable Hamiltonian we can now search for the the corresponding deformed Lagrangian of the Yang-Mills theory, such that when the deformation parameter q is sent to 1 we recover $N = 4$ theory. Let us note that each term in the spin chain Hamiltonian comes from the quartic term in the Lagrangian. For example the coefficient in front of the $E_{il}E_{jk}$ term of the Hamiltonian corresponds to the following vertex interaction

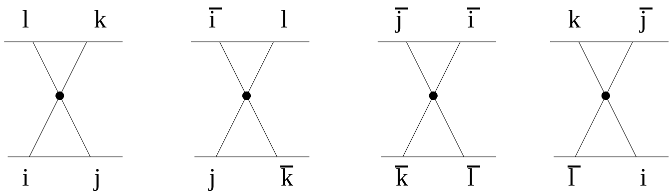


which comes from the term $\mathrm{tr}\Phi_i\Phi_j\bar{\Phi}_k\bar{\Phi}_l$ of the Lagrangian. Since the trace is cyclic

$$
\mathrm{tr}\Phi_i\Phi_j\bar{\Phi}_k\bar{\Phi}_l = \mathrm{tr}\Phi_j\bar{\Phi}_k\bar{\Phi}_l\Phi_i = \mathrm{tr}\bar{\Phi}_k\bar{\Phi}_l\Phi_i\Phi_j = \mathrm{tr}\bar{\Phi}_l\Phi_i\Phi_j\bar{\Phi}_k, \tag{6.20}
$$

the following diagrams

[4] A less cryptic form of this Hamiltonian can be found in the Appendix.



have equal contributions. It follows that if the theory admits a deformation that respects the $SO_q(6)$ symmetry, then, for example, the terms $E_{ii} \otimes E_{\bar{i}\bar{i}}$ and $E_{\bar{i}\bar{i}} \otimes E_{ii}$ have to have exactly the same coefficients. For concreteness let us presume $i < \bar{i}$, i.e. $i \leq 3$. Inspecting Eq.(6.18) the corresponding terms are

$$\left((q+q^{-1})^2 + 1 + q^{-2} + q^{2i-4}\right) E_{ii} \otimes E_{\bar{i}\bar{i}}, \tag{6.21}$$

and

$$\left((q+q^{-1})^2 + 1 + q^{2} + q^{4-2i}\right) E_{\bar{i}\bar{i}} \otimes E_{ii}, \tag{6.22}$$

with $i = 1, 2$, or 3. It is clear that these terms do not satisfy the cyclicity condition for $i \neq 3$ unless $q^4 = 1$. These are exactly the cases that correspond to orbifolds if q is real. For q complex however the Hamiltonian is not unitary.

Examining the terms $X\bar{X}\hat{X}\hat{\bar{X}}$ and $\bar{X}X\hat{\bar{X}}\hat{X}$ in the general $SO_q(6)$ symmetric form of the Hamiltonian Eq.(6.4) one can verify that the Hamiltonian does not satisfy the cyclicity property.

We can conclude that the XXZ $SO(6)$ spin chain does not correspond to any Lagrangian deformation of the $N = 4$ Yang-Mills theory which is due solely to a change in the scalar potential. Moreover, thanks to the analysis of subsection 6.1, we conclude that none of the spin chains with $SO_q(6)$ symmetry comes from a Lagrangian.

7. Correspondence between integrable spin chains and matrix quantum mechanics

So far we have presented results that show that for the most part integrability is very closely related to the $AdS_5 \times S^5$ geometry. Indeed, all the examples that we showed that were integrable were not too different from orbifold geometries, and can be even considered as limits of orbifold geometries. Then we showed that it was not possible to deform the full spin chain model and retain the conformal invariance of the theory, indeed it was not even possible to deform the potential so that the spin chain would have $SO_q(6)$ symmetry.

In this section we will show that there is a modification that resolves this issue in order to obtain an integrable theory with different structure. However, the deformation will involve terms in the effective action for the theory on S^3 which will

have derivative interactions and therefore will not respect the conformal invariance of the theory. This is a generalization of the idea of Roiban [37] of matching spin chains to field theories. Notice that in the previous section we found an obstruction to have this realized in a simple fashion. In the work of Roiban only subsectors were considered, and within those subsectors one could sometimes obtain integrability. Essentially we shall construct a matrix model that corresponds to the XXZ SO(6) spin chains obtained in this paper. Equivalence between integrable XXX spin chains and matrix models was explored, for example, in [38] where Poisson structure and conserved charges are described and matched. [38] also defines an interesting 'classical limit' of the spin chain. One might explore implications of this work to various XXZ spin chains and study their classical limits. Also, a matrix model associated to the eleventh-dimensional plane wave geometry has been studied from the large N integrability point of view in [39], where an analysis was done up to three loops in a particular subsector.

So far we have concentrated on the operators made out of the scalars of the theory. In the operator state correspondence these only involve the lowest spherical harmonic of the fields on the S^3. [5] Indeed, we can think of each of them as a matrix quantum mechanical degree of freedom. Under this identification, to each possible state on a single site in the spin chain Hamiltonian there corresponds one matrix quantum mechanical degree of freedom. The operators with derivatives fill the infinite dimensional unitary representation of $SU(2,2|4)$ (the singleton representation). From this point of view, it is better to start more simply with spin chain models which have only a finite number of states at each site, and deal with just a finite number of matrices.

So let us begin with this setup, and consider a system consisting of a collection of k $N \times N$ Hermitian matrices M_j, whose free Lagrangian is given by

$$L = \sum_j N(\frac{1}{2}\mathrm{tr}(DM_j^2) + \frac{1}{2}\mathrm{tr}(M_j^2)) \tag{7.1}$$

where $DM_j = \dot{M}_j - i[M_j, A_0]$ and A_0 is a Hermitian matrix. This system has gauge invariance under $SU(N)$ time dependent gauge transformations where all matrices transform by conjugation and A_0 is an $SU(N)$ connection. It is also a Lagrange multiplier whose equations of motion imply that the states of the theory carry no $SU(N)$ angular momentum.

When we quantize the theory we can choose the gauge $A_0 = 0$ and impose the gauge constraint on the states, so that the states of the theory have to be represented as singlets of the $SU(N)$ group.

[5]Remember that the higher spherical harmonics are obtain by acting with derivatives, as these are the ones that carry the quantum numbers under the $SO(4)$ subgroup of $SO(4,2)$ which are realized as isometries of S^3.

Since the theory with $A_0 = 0$ is a free theory, we can write the quantum Hamiltonian in terms of creation and annihilation operators, so that for each pair of matrices $M_i, P_i = N\dot{M}_i$ we have a matrix of creation and annihilation operators, and then the Hamiltonian of the quantum system is given by (up to some trivial normal ordering constant)

$$H = \sum_s \mathrm{tr}(a_s^\dagger a_s) \tag{7.2}$$

Each of the matrix components of $a_s^\dagger$ is an individual oscillator, with one upper index of $SU(N)$ and one lower index of $SU(N)$. Moreover all creation (annihilation) operators commute with all creation (annihilation) operators, and we have the commutation relation

$$[(a_s^\dagger)^i_j, (a_t)^l_m] = \delta_{st}\delta^i_m\delta^l_j \tag{7.3}$$

A state of the theory (without imposing the gauge constraint) is given by acting with an arbitrary number of creation operators on a vacuum defined by $a|0>= 0$ for all possible annihilation operators.

If we act with n creation operators, then the collection of these states transforms under the $SU(N)$ group as members of representations with n upper $SU(N)$ indices and n lower $SU(N)$ indices.

In the gauged quantum theory we need to impose the gauge constraint which forces us onto a singlet of the $SU(N)$ group. This condition implies that every upper index is contracted with a lower index and summed over the indices.

We can collect the contractions of gauge indices by using matrix multiplication, as each upper index is contracted with one particular lower index.

This looks like a product of traces of creation operators

$$|\psi> \sim \mathrm{tr}(a^\dagger_{s_{1,1}} a^\dagger_{s_{2,1}} \dots a^\dagger_{s_{k_1,1}})\mathrm{tr}(a^\dagger_{s_{2,2}} \dots a^\dagger_{s_{k_2,2}}) \dots \tag{7.4}$$

where the indices indicate a collection of m traces of lengths $k_1, k_2, \dots k_m$ with $\sum k_i = n$.

For each trace we associate the list of labels of the creation operators in the order that they appear in the matrix

$$\mathrm{tr}(a^\dagger_{s_1} \dots a^\dagger_{s_t}) \sim (s_1, \dots, s_t) \tag{7.5}$$

and such an ordered list of labels is a word, where the letters are the labels of the matrices. Words related by cyclic permutations are equivalent, as they can be obtained from each other by commuting the creation operators past each other and at the end combining the gauge indices as matrix multiplication.

It is a non-trivial fact that in the large N limit, $N \to \infty$, when we keep n finite, the multi-trace states describe an approximate Fock space of states, where there is one oscillator per cyclic word. These statements reflect the large N combinatorics of free field contractions, and the $1/N$ expansion for the normalization of the states.

Different states in this Fock space described above are only approximately orthogonal, as their overlap amplitudes are suppressed by inverse powers of N.

Given k matrices, the number of cyclic words of length n is roughly of order k^n/n, which grows exponentially in n. Thus, although there are only finitely many single trace states with energy less than n, the number of such states grows exponentially. When one considers that one has a Fock space with an exponentially growing number of oscillators, then the entropy grows exponentially in n.

This growth holds so long as $n << N$, so we are taking first the limit $N \to \infty$ and then n large. The order of limits matters in this case, because the matrix quantum mechanics for finite N is given by N^2 free oscillators. The number of oscillators accessible does not grow with the energy for energies that are very large.

Now, we want to consider perturbing the matrix Hamiltonian in the large N limit by using a single trace polynomial in the matrices and their derivatives. The restriction to planar diagrams will give us interactions on the states which are local on the words. If one ignores the cyclic condition, one can represent it by acting on chains of letters by a local Hamiltonian. Generically this will produce a theory where the time evolution changes the number of letters in a word. However, if one uses perturbation theory, then as a first step we need to calculate the expectation value of the Hamiltonian in the energy basis. This is a degenerate perturbation theory, but the number of oscillators will stay fixed.

The basic point of the correspondence we want to make is that the set of single trace states (which we call single string states) is labelled by words, which is the same type of labelling that takes place for a spin chain. Given a spin chain model, the dynamics is usually encoded in the Hamiltonian that describes the time evolution of the spin chain model.

Now we want to perturb the Hamiltonian of the free matrix model to get a match with a local Hamiltonian for the spin chain model, rather than working in the Lagrangian formalism. This can be considered either as the first term in a perturbation theory expansion, or we might fabricate a theory which preserves the length of the chain automatically. Either way, we will obtain (effective) Hamiltonians which preserve the length of the chain. Now we want to consider how it relates to a spin chain model.

The Hamiltonian of the spin chain will consist on a collection of terms which are nearest neighbor, next to nearest neighbor, etc. These form a collection of maps from $H_2 : V \otimes V \to V \otimes V$, $H_3 : V \otimes V \otimes V \to V \otimes V \otimes V$ etc.

We can use a tensor notation as follows

$$H_2(v, w) = H^{\nu_1\nu_2}_{\mu_1\mu_2} v^{\mu_1} w^{\mu_2} \tag{7.6}$$

$$H_3(u, v, w) = H^{\nu_1\nu_2\nu_3}_{\mu_1\mu_2\mu_3} u^{\mu_1} v^{\mu_2} w^{\mu_3} \tag{7.7}$$

where the indices refer to a particular basis for the vector spaces V.

The Hamiltonian of the spin chain will be given by

$$H = \sum_i H_1^i + \sum_i H_2^{i,i+1} + \sum_i H_3^{i,i+1,i+2} + \dots \tag{7.8}$$

where the index on the individual terms tells us which lattice sites transform under the action of each term in the Hamiltonian.

Now, consider the perturbation of the matrix model Hamiltonian which is given by

$$\delta H_2 = \frac{\lambda_1}{N} H^{\nu_1\nu_2}_{\mu_1\mu_2} \mathrm{tr}(a^\dagger_{\nu_1} a^\dagger_{\nu_2} a_{\mu_2} a_{\mu_1}) \tag{7.9}$$

It is easy to check that this Hamiltonian acts on the matrix model words exactly as the Hamiltonian H_2 does on the spin chain when one considers only the planar contribution. The normalization with N is dictated by the planar diagram expansion.

Similarly, one can check that

$$\delta H_3 = \frac{\lambda_2}{N^2} H^{\nu_1\nu_2\nu_3}_{\mu_1\mu_2\mu_3} \mathrm{tr}(a^\dagger_{\nu_1} a^\dagger_{\nu_2} a^\dagger_{\nu_3} a_{\mu_3} a_{\mu_2} a_{\mu_1}) \tag{7.10}$$

acts in the same way as the Hamiltonian H_3. This can be generalized to any local interaction on the spin chain model: for terms involving n nearest neighbors, we will have a trace with n creation operators followed by n annihilation operators in that order. Other types of orderings will produce (at least at leading order) subleading terms in the $1/N$ expansion and will be dropped. These might appear at higher loops and contribute as much as planar interactions.

From this it is clear that given any spin chain model, one can find a matrix model which realizes it, with one matrix for each state of the local spin variable on the spin chain. If one uses infinite dimensional representations for each site, one would want to avoid the infinite degeneracy of the spin chain, so the term H_1 can be modified so that different states have different bare energy. Also, one would have an infinite number of matrix degrees of freedom. However, these could result from compactifications of higher dimensional gauge theories on some manifold, e.g, $\mathcal{N} = 4$ SYM on S^3. Then the bare energies of the sites would encode the harmonic analysis of the theory.

Notice that the above prescription uses creation and annihilation operators in equal numbers, so that the length of the chain is preserved. Also, when these appear, in general, one has terms in the Hamiltonian which are polynomial in the matrices and the momenta together, and of order higher than 2. Polynomials in the matrices alone will also lead to terms with only creation or annihilation operators which would change the length of the chain. These don't contribute to the spectrum of the Hamiltonian in perturbation theory to leading order because they mix states with different free energy, but they generically contribute at higher orders. These are absent in the above description, so it is inevitable to introduce the conjugate

momenta to the matrices M if we want to insist on the exact form of the Hamiltonian as above and not just as an approximation. Also, from the obstruction found in the previous section we realize that generically these higher derivative terms are unavoidable if we want to match to a given Hamiltonian. This leads to Hamiltonians with higher derivative terms, and hence these also affect the Lagrangian formulation of the theory. From the point of view of $\mathcal{N}=4$ SYM what we see is that the price to pay for deforming the $SO(6)$ to obtain $SO_q(6)$ symmetry is that we have to give up the conformal invariance, and we even have to give up the renormalizability by power counting of the associated field theory.

Notice that the generalized form of the deformed Hamiltonian for a finite number of matrices we have built commutes with the total particle number, which is a strictly positive integer number. This suggests that the theory above might be describing partons of fixed light-cone momentum. From the holographic point of view this might be the type of the matrix model that describes light-cone quantization of some geometries. These are very different in character from theories where the partons come from the rank of the gauge group.

Indeed, this seems to be the natural setup to look for the holographic dual of the maximally supersymmetric plane wave in ten dimensions, as it is known that the conformal boundary of the geometry is one dimensional [46].

A second point that should be noticed is that in the large N limit, most of the occupation numbers of oscillators are 0 or 1, as the total number of oscillators in the theory is of order N^2 and one works at small occupation numbers. Thus these occupation numbers for practical purposes satisfy Fermi statistics. Indeed, one can generalize these models to theories with fermion oscillators instead of bosons. The main difference will be realized in the properties that have to do with cyclicity of the trace, because there would be additional minus signs after permuting the operators. This translates onto a spin chain model with different boundary conditions and the possibility of introducing string states which obey Fermi statistics. It would be natural if one is to try to understand supersymmetric matrix models.

Now, returning to the deformations of $\mathcal{N}=4$ SYM, it seems that if we are willing to do away with conformal symmetry, it is more natural to extend the quantum deformation to a full $SU_q(2,2|4)$ symmetry, which will also remove the invariance under the $SO(4,2)$, but that is still tractable with the help of the symmetry considerations.

Indeed, let us consider this possibility in the following. Remembering the general discussion that led to equation (6.4), in terms of quantum group representations, it is better to write the Hamiltonian as

$$aP_{20} + (a+c)P_{15} + (b+a)P_s \tag{7.11}$$

for the $SO(6)$ scalar sector. We can always shift the value of a so that it vanishes, $a=0$, so we will choose the coefficient of P_{20} to be zero. This choice reflects the

triviality of a deformation which is proportional to the identity. It is also the choice that leads to having protected BPS operators in the final theory.

The representation content of the full $SU(2,2|4)$ product of two singletons has the same multiplication table as two unitary representations of $SL(2,R)$

$$V_F \otimes V_F \sim V_0 \oplus V_1 \oplus V_2 \oplus \ldots \tag{7.12}$$

And the 20 of $SO(6)$ is a primary field in V_1, the 15 is a primary field in V_2 and the singlet is a primary field in V_3. The higher order operators appearing after the $\ldots$ all involve derivatives, and these have not been part of the discussion as of yet. However, the structure of the integrable Hamiltonian for the $SU_q(2,2|4)$ deformation should follow the same decomposition principle, so the general spin chain Hamiltonian has to have the form

$$H \sim \sum_{i=1}^{\infty} a_i P_i \tag{7.13}$$

where the a_i are some numbers determined by integrability. We can always choose $a_1 = 0$, and this condition determines the rest of H up to a constant multiplicative factor. Indeed, when $q = 0$ these are the harmonic numbers $a_0 = 0, a_1 = 1, \ldots, a_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$ [7, 8]. The R-matrix $R(u)$ will also have the same form, where the a_i will be functions of the spectral parameter u. In the case we studied, let $s = \exp(2u)$, and the R-matrix for the $SU_q(4)$ spin chain can be written in a normalization such that

$$R(s) = P_{20} + \frac{q^2 - s}{1 - q^2 s} P_{15} + \frac{(q^2 - s)(q^4 - s)}{(1 - q^2 s)(1 - q^4 s)} P_1 \tag{7.14}$$

noting the similarity of the multiplication table of two singletons of $SU(2,2|4)$ to representations of $SL(2,R)$, we can guess the form of the associated R-matrix by following the known result for $SL(2,R)$, see for example [47]. We obtain that

$$R(s) = \sum_{i=0}^{\infty} r_i(s) P_i \tag{7.15}$$

is given by

$$r_i(s) = \prod_{m=1}^{i} \frac{1 - q^{2m} s}{q^{2m} - s} \tag{7.16}$$

¿From here we get

$$a_i = -(1 - q^2) \frac{d}{ds} r_i(s)|_{s=1} = \sum_{m=1}^{i} \frac{1 + q^{2m}}{(q^{2m} - 1)/(q^2 - 1)} \tag{7.17}$$

where the factor of $1 - q^2$ is inserted so that we have a nice $q \to 1$ limit. It is easy to see that the denominators above are proportional to the harmonic numbers in the limit $q \to 1$, so the result coincides in the limit with $\mathcal{N} = 4$ SYM.

8. Conclusion

In this paper we have tried to generalize the integrability of the one-loop planar matrix of anomalous dimensions for $\mathcal{N} = 4$ SYM theory to other theories with less supersymmetry, by trying to understand the simplest marginal deformations of $\mathcal{N} = 4$ SYM with a lot of symmetry (we require to keep the Cartan subgroup of the R-symmetry). Our results show that certain deformations of $\mathcal{N} = 4$ that interpolate between $\mathcal{N} = 4$ and it's orbifolds with discrete torsion are integrable. The orbifolds themselves were expected to be locally integrable, and indeed, we have shown that all of these lead to the same spin chain Hamiltonian as the $\mathcal{N} = 4$ SYM theory, but with twisted boundary conditions. We gave a very detailed form of the twisting boundary conditions for subsectors of the theory. More generally, we studied the q-deformation of $\mathcal{N} = 4$ SYM theory and showed that generically it is not integrable, although various subsectors of it are integrable.

We tried to generalize the deformations further by keeping a larger integrable subsector related to $SU_q(3)$ symmetry while preserving the four dimensional conformal group. We showed that this deformation was likely not to be fully integrable because it was impossible to generalize it further to an $SU_q(4)$ symmetry, which would be the quantum group remnant of the $SO(6)$ group of R-symmetries of the original $\mathcal{N} = 4$ SYM theory. For this last part we computed the full Hamiltonian of the quantum spin chain with spins in the 6 dimensional representation of $SO_q(6)$. We did this by computing the spectral R matrix in the antisymmetric of $SU(F)$ for any F by fusion of representations, and in particular for $F = 4$ it produced to the result we needed.

We have found in the course of our investigations that the constraints imposed by integrability are very hard to meet and that it is very remarkable that there is a theory in four dimension which displays integrability at all.

In the course of generalizing the relations between field theories and integrable spin chains we also discovered that the natural setup for analyzing these questions is in terms of multi-matrix quantum mechanics. Here we found that it is possible to obtain a correspondence between arbitrary spin chains (both integrable and not) and the large N limit of multi-matrix quantum mechanics. We hope that these ideas might serve to produce holographic duals of plane wave geometries.

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Appendix

Fundamental representation of $SO_q(6)$

$$
\begin{aligned}
F_1 v_2 = v_3 \ &, \ E_1 v_3 = v_2 \\
F_1 v_4 = v_5 \ &, \ E_1 v_5 = v_4 \\
F_2 v_1 = v_2 \ &, \ E_2 v_2 = v_1 \\
F_2 v_5 = v_6 \ &, \ E_2 v_6 = v_5 \\
F_3 v_2 = v_4 \ &, \ E_3 v_4 = v_2 \\
F_4 v_3 = v_5 \ &, \ E_3 v_5 = v_3
\end{aligned}
$$

	v_1	v_2	v_3	v_4	v_5	v_6
H_1	0	1	−1	1	−1	0
H_2	1	−1	0	0	1	−1
H_3	0	1	1	−1	−1	0

Orthonormal basis of 15 of $SU_q(4)$

	H_1	H_2	H_3
$w_1 = (v_1v_2 - qv_2v_1)/\sqrt{1+q^2}$	1	0	1
$w_2 = (v_1v_3 - qv_3v_1)/\sqrt{1+q^2}$	−1	1	1
$w_3 = (v_1v_4 - qv_4v_1)/\sqrt{1+q^2}$	1	1	−1
$w_4 = (v_2v_3 - qv_3v_2)/\sqrt{1+q^2}$	0	−1	2
$w_5 = (v_1v_5 - qv_5v_1)/\sqrt{1+q^2}$	−1	2	−1
$w_6 = (v_2v_4 - qv_4v_2)/\sqrt{1+q^2}$	2	−1	0
$w_7 = (v_3v_4 - v_4v_3)/\sqrt{2}$	0	0	0
$w_8 = (v_1v_6 - v_6v_1 + q^{-1}v_2v_5 - qv_5v_2)/(q+q^{-1})$	0	0	0
$w_9 = \sqrt{\frac{2}{q^2+q^{-2}}}\left(v_1v_6 - v_6v_1 - qv_2v_5 + q^{-1}v_5v_2 + \frac{q^2-q^{-2}}{2}(v_3v_4+v_4v_3)\right)/(q+q^{-1})$	0	0	0
$w_{10} = (v_4v_5 - qv_5v_4)/\sqrt{1+q^2}$	0	1	−2
$w_{11} = (v_2v_6 - qv_6v_2)/\sqrt{1+q^2}$	1	−1	1
$w_{12} = (v_3v_5 - qv_5v_3)/\sqrt{1+q^2}$	−1	1	0
$w_{13} = (v_4v_6 - qv_6v_4)/\sqrt{1+q^2}$	1	−1	−1
$w_{14} = (v_3v_6 - qv_6v_3)/\sqrt{1+q^2}$	−1	−1	1
$w_{15} = (v_5v_6 - qv_6v_5)/\sqrt{1+q^2}$	−1	0	−1

$SO_q(6)$ **XXZ Hamiltonian**

$$H = \frac{q^2 - q^{-2}}{2} \left(\frac{d}{du} R(u) \right)\bigg|_{u=0} R(0)^{-1} =$$

$E_{ii}E_{ii}$ Terms

$$= (q+q^{-1})^2 (E_{1,1} \otimes E_{1,1} + E_{2,2} \otimes E_{2,2} + E_{3,3} \otimes E_{3,3} + E_{\bar{1},\bar{1}} \otimes E_{\bar{1},\bar{1}} + E_{\bar{2},\bar{2}} \otimes E_{\bar{2},\bar{2}} + E_{\bar{3},\bar{3}} \otimes E_{\bar{3},\bar{3}}) -$$

$E_{ij}E_{ji}$ with $j \neq i, \bar{i}$ Terms

$$\begin{aligned} -(q+q^{-1}) \ (&E_{1,2} \otimes E_{2,1} + E_{1,3} \otimes E_{3,1} + E_{1,\bar{2}} \otimes E_{\bar{2},1} + E_{1,\bar{3}} \otimes E_{\bar{3},1} + E_{2,1} \otimes E_{1,2} + \\ &+ E_{2,3} \otimes E_{3,2} + E_{2,\bar{1}} \otimes E_{\bar{1},2} + E_{2,\bar{3}} \otimes E_{\bar{3},2} + E_{3,1} \otimes E_{1,3} + E_{3,2} \otimes E_{2,3} + \\ &+ E_{3,\bar{1}} \otimes E_{\bar{1},3} + E_{3,\bar{2}} \otimes E_{\bar{2},3} + E_{\bar{1},2} \otimes E_{2,\bar{1}} + E_{\bar{1},3} \otimes E_{3,\bar{1}} + E_{\bar{1},\bar{2}} \otimes E_{\bar{2},\bar{1}} + \\ &+ E_{\bar{1},\bar{3}} \otimes E_{\bar{3},\bar{1}} + E_{\bar{2},1} \otimes E_{1,\bar{2}} + E_{\bar{2},3} \otimes E_{3,\bar{2}} + E_{\bar{2},\bar{1}} \otimes E_{\bar{1},\bar{2}} + E_{\bar{2},\bar{3}} \otimes E_{\bar{3},\bar{2}} + \\ &+ E_{\bar{3},1} \otimes E_{1,\bar{3}} + E_{\bar{3},2} \otimes E_{2,\bar{3}} + E_{\bar{3},\bar{1}} \otimes E_{\bar{1},\bar{3}} + E_{\bar{3},\bar{2}} \otimes E_{\bar{2},\bar{3}}) \end{aligned}$$

$E_{ii}E_{jj}$ with $j \neq i, \bar{i}$ Terms

$$\begin{aligned} +(q+q^{-1}) \ \big(&(q+2q^{-1})E_{1,1} \otimes E_{2,2} + (q+2q^{-1})E_{1,1} \otimes E_{3,3} + (q+2q^{-1})E_{1,1} \otimes E_{\bar{2},\bar{2}} + \\ &+ (q+2q^{-1})E_{1,1} \otimes E_{\bar{3},\bar{3}} + (2q+q^{-1})E_{2,2} \otimes E_{1,1} + (q+2q^{-1})E_{2,2} \otimes E_{3,3} + \\ &+ (q+2q^{-1})E_{2,2} \otimes E_{\bar{1},\bar{1}} + (q+2q^{-1})E_{2,2} \otimes E_{\bar{3},\bar{3}} + (2q+q^{-1})E_{3,3} \otimes E_{1,1} + \\ &+ (2q+q^{-1})E_{3,3} \otimes E_{2,2} + (q+2q^{-1})E_{3,3} \otimes E_{\bar{1},\bar{1}} + (q+2q^{-1})E_{3,3} \otimes E_{\bar{2},\bar{2}} + \\ &+ (2q+q^{-1})E_{\bar{1},\bar{1}} \otimes E_{2,2} + (2q+q^{-1})E_{\bar{1},\bar{1}} \otimes E_{3,3} + (2q+q^{-1})E_{\bar{1},\bar{1}} \otimes E_{\bar{2},\bar{2}} + \\ &+ (2q+q^{-1})E_{\bar{1},\bar{1}} \otimes E_{\bar{3},\bar{3}} + (2q+q^{-1})E_{\bar{2},\bar{2}} \otimes E_{1,1} + (2q+q^{-1})E_{\bar{2},\bar{2}} \otimes E_{3,3} + \\ &+ (q+2q^{-1})E_{\bar{2},\bar{2}} \otimes E_{\bar{1},\bar{1}} + (2q+q^{-1})E_{\bar{2},\bar{2}} \otimes E_{\bar{3},\bar{3}} + (2q+q^{-1})E_{\bar{3},\bar{3}} \otimes E_{1,1} + \\ &+ ((2q+q^{-1})E_{\bar{3},\bar{3}} \otimes E_{2,2} + (q+2q^{-1})E_{\bar{3},\bar{3}} \otimes E_{\bar{1},\bar{1}} + (q+2q^{-1})E_{\bar{3},\bar{3}} \otimes E_{\bar{2},\bar{2}}\big) + \end{aligned}$$

$E_{ii}E_{\bar{i}\bar{i}}$ Terms

$$\begin{aligned} &+(q^2+3+3q^{-2})E_{1,1} \otimes E_{\bar{1},\bar{1}} + (q^2+4+2q^{-2})E_{2,2} \otimes E_{\bar{2},\bar{2}} + (2q^2+3+2q^{-2})E_{3,3} \otimes E_{\bar{3},\bar{3}} + \\ &+(3q^2+3+q^{-2})E_{\bar{1},\bar{1}} \otimes E_{1,1} + (2q^2+4+q^{-2})E_{\bar{2},\bar{2}} \otimes E_{2,2} + (2q^2+3+2q^{-2})E_{\bar{3},\bar{3}} \otimes E_{3,3} - \end{aligned}$$

$E_{i\bar{i}}E_{\bar{i}i}$ Terms

$$-E_{1,\bar{1}} \otimes E_{\bar{1},1} - E_{2,\bar{2}} \otimes E_{\bar{2},2} - E_{3,\bar{3}} \otimes E_{\bar{3},3} - E_{\bar{1},1} \otimes E_{1,\bar{1}} - E_{\bar{2},2} \otimes E_{2,\bar{2}} - E_{\bar{3},3} \otimes E_{3,\bar{3}} -$$

$E_{ij}E_{\bar{i}\bar{j}}$ with $j \neq i, \bar{i}$ Terms

$$
\begin{aligned}
&-q^{-1}E_{1,2}\otimes E_{\bar{1},\bar{2}}+E_{1,3}\otimes E_{\bar{1},\bar{3}}-qE_{1,\bar{2}}\otimes E_{\bar{1},2}+E_{1,\bar{3}}\otimes E_{\bar{1},3}-q^{-1}E_{2,1}\otimes E_{\bar{2},\bar{1}}-qE_{2,3}\otimes E_{\bar{2},\bar{3}}-\\
&-q^{-1}E_{2,\bar{1}}\otimes E_{\bar{2},1}-qE_{2,\bar{3}}\otimes E_{\bar{2},3}+E_{3,1}\otimes E_{\bar{3},\bar{1}}-qE_{3,2}\otimes E_{\bar{3},\bar{2}}+E_{3,\bar{1}}\otimes E_{\bar{3},1}-q^{-1}E_{3,\bar{2}}\otimes E_{\bar{3},2}-\\
&-q^{-1}E_{\bar{1},2}\otimes E_{1,\bar{2}}+E_{\bar{1},3}\otimes E_{1,\bar{3}}-qE_{\bar{1},\bar{2}}\otimes E_{1,2}+E_{\bar{1},\bar{3}}\otimes E_{1,3}-qE_{\bar{2},1}\otimes E_{2,\bar{1}}-q^{-1}E_{\bar{2},3}\otimes E_{2,\bar{3}}-\\
&-qE_{\bar{2},\bar{1}}\otimes E_{2,1}-q^{-1}E_{\bar{2},\bar{3}}\otimes E_{2,3}+E_{\bar{3},1}\otimes E_{3,\bar{1}}-qE_{\bar{3},2}\otimes E_{3,\bar{2}}+E_{\bar{3},\bar{1}}\otimes E_{3,1}-q^{-1}E_{\bar{3},\bar{2}}\otimes E_{3,2}.
\end{aligned}
$$

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